Chapter6 – Filter Design

In this chapter

- Filter Terminologies
- Filter Design methods
- Analog Filters

Introduction

A ‘Signal’ is any useful information and a ‘Noise’ is the not so useful information, invariably, a part of the Signal. The Signals, seemingly arbitrary in nature, may be thought of as superimposed sinusoidal functions (the fundamental and the harmonics), as described by Fourier. You can imagine what would happen to Signal frequencies, if they were passed through a system of energy storage elements (such as the Linear Time Invariant systems we have been studying, that in itself has an impulse response in the form of sinusoidal functions), obviously, the matching frequencies in the Signal and System would be enhanced and the unmatched frequencies would be subsided. The result is like a filter, separating the desired frequencies from the undesired. We can tailor the response of such systems to our wishes, by physically choosing the components (in case of an analog system) or the coefficients (in case of a digital system) and this is the basics of the filter design, suppressing or enhancing a specific frequency or a set of frequencies by passing a Signal source through a system of energy storage elements. Filters modify signals in way specified by the filter’s frequency response.

In essence, a filter is a system, capable of generating response based on the frequency selection. An analog filter is an electrical network comprising physical components of the resistors, capacitors and inductors, whereas, a digital filter is a computer software algorithm producing the filter response. Filters can either extract and amplify or reject and subside a specific frequency or range of frequencies, but they can also be designed to create a pattern of frequencies as a response to a stimulus. The current trend is towards the Digital Filter design but still, the analog filters are much in demand in high frequency arena as well as feedback and control systems where the response time is critical.

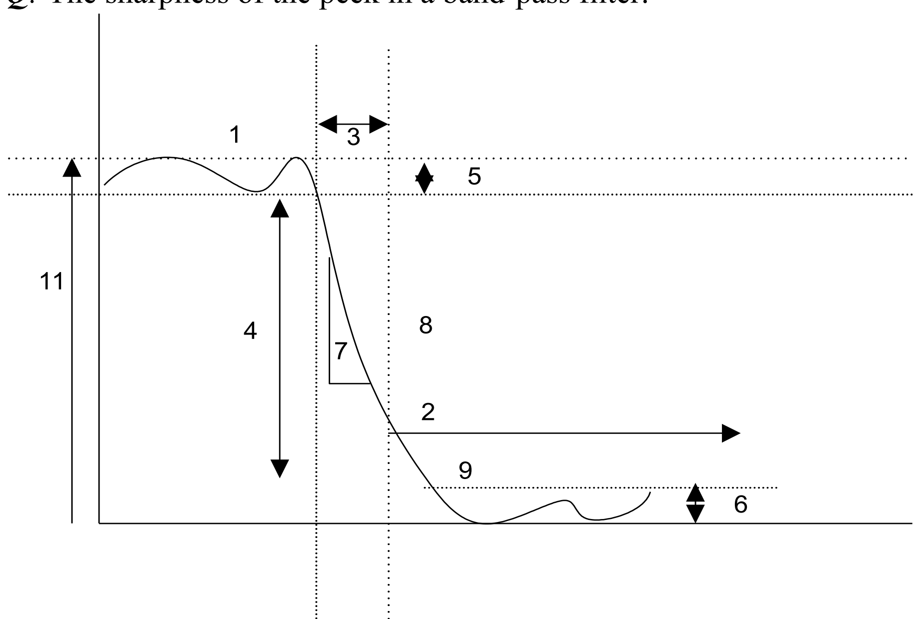
We have discussed filters in our previous chapter of solutions of differential equations and have designed some simple low-pass and high-pass filters, using both, the analog and the digital techniques. The reason for a tandem discussion is that most analog filters are a precursor of digital filters and have roots behind the filter designs mathematics. The criteria of any filter design are to establish the boundary of the desirable frequencies and then define how much suppression of the undesired frequencies is acceptable. Remember, frequencies can never be eliminated they can only be suppressed. The first criterion provides the cutoff frequency and the second one establishes how many poles and zeros are required in the filter design. The design methods are well established in both, the analog and the digital domain and the design procedures are a simple matter of selecting the appropriate components or coefficients. In this chapter, some commonly used designs of analog filters are being discussed that may become prototype for our next chapter of the Digital Filter design.

We begin our discussion with the explanation of terms, normally used in both, the analog and the digital filters.

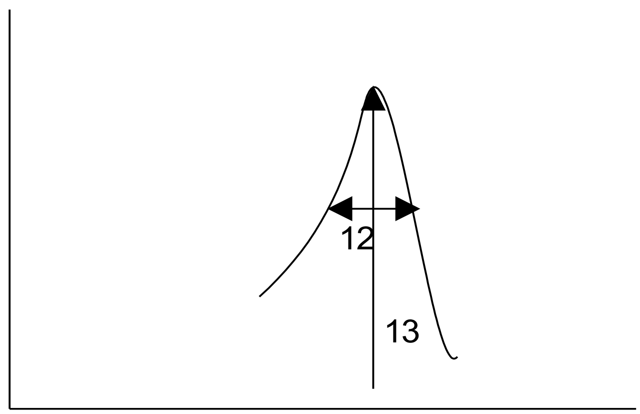
Filter Terminologies

The following terminologies explain different aspects of filter characteristics. See Figure 6.1.a and 6.1.b for a reference.

1. **Pass-band**: Frequency range preserved in the output
2. **Stop-band**: Frequency range suppressed in the output
4. **Gain**: Amount of maximum amplification in the output
6. **Transition-band**: The region of frequencies between pass-band and stop-band.
7. **Stop-band attenuation**: The difference in dB between the pass-band and the stop-band gain.
8. **Pass-band ripple**: The maximum fluctuation in the frequency response in the pass-band.
9. **Stop-band ripple**: The maximum fluctuation in the frequency response in the stop-band.
10. **Roll off rate**: The steepness of the slope in the transition band, (multiples of 20db/decade).
11. **Order**: The number of poles in the system function *H(s).* The higher the order, the steeper the roll off
12. **Cutoff frequency**: The edge of the pass-band. (3db point)
13. ***Q:*** The sharpness of the peek in a band-pass filter.



******Insert Figure 6.1a here*******
Figure 6.1a Different characteristics of a filter response



****Insert Figure 6.1b here*****
Figure 6.1 The cutoff frequency and Q of a band-pass filter

Filter Design methods

We have seen in the previous chapter of the Solutions of Differential Equations, how a first and second order differential equation's response acts as a frequency selector. The filter design is essentially implementing solutions of differential equations, either through the Laplace Transform or the Convolution process. but the modeling is done through the Transfer Function that formed the input and output relationship.

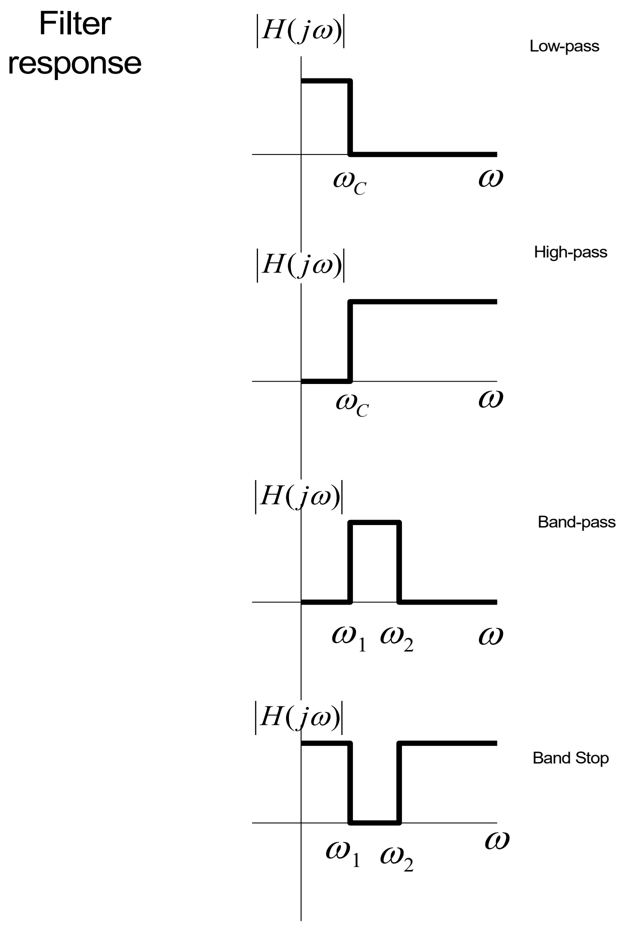
The higher frequencies are suppressed from the voltage across the capacitor, in a network of resistor and capacitor. Similarly, the lower frequencies are suppressed from the voltage across the resistor, in the network of a resistor and capacitor and a band-limited response is achieved with a second order section.

The concept of poles and zeros is central to the filter design as they can be directly translated to the Transfer Function of the system. You have seen in the previous chapter, how magnitude of pole and zero vectors define the cutoff regions. The frequencies beyond the pole frequencies are reduced substantially and the frequencies beyond the zero vectors are enhanced largely. Being a linear system, the overall filter response is the sum of the individual poles and zeros response. Although, any form and shape of the frequency response could be obtained by carefully selecting these vectors, but the response generally falls into the following four basic types,

Low-pass, high-pass, band-pass and **band-stop** filters. There are variations of filters such as notch filters, narrow band-pass etc. that are special implementations of the band-pass and the band-stop filters.

As the name suggest, the **low-pass** filters allow low-frequency to pass through, but stop high frequencies, the **high-pass** filters allow high frequencies to pass through but stop low frequencies, the **band-pass** filters allow a specific band of frequencies to pass through but stop all others and the **band-stop** filters allow all frequencies to pass through except a certain band.

The fact that the LTI systems only alter the amplitude or phase but not the frequency itself, leaves us with only one choice and that is to suppress the undesired frequencies. Ideally, we would like to have the desired frequencies untouched while the undesired frequencies eliminated, as shown in the Figure 6.2, but this is not practical as it requires infinite filter blocks, so we will leave the quest for achieving the ideal response only as a goal to strive.



******Insert Figure 6.2 here*******

Figure 6.2. Ideal filter response: a) Low pass, b) high pass, c) band pass d) band stop filters

The digital and analog filters may be identical in their response, but the implementation of each is entirely a different matter. For analog filters, the Transfer Function is implemented as a circuit design with the coefficient values translated to the component values of the resistors, capacitors and inductors (op-amps may be added in place of inductors and to isolate the blocks of circuits from one another). On the other hand, the digital filters are realized by transforming the Transfer Function in to a difference equation that is to be solved with convolution using an iterative algorithm.

Every filter has a corresponding Transfer Function, but it is easier to implement filters in terms of the building blocks of Transfer Functions. It not only reduces the complexity of the overall system, but also gives a modular approach to the implementation process. The building block approach is explained in the next section.

Building Blocks of Transfer Functions

We have seen in the previous chapter, that each pole contributes to a 20 dB drop in the roll-off rate of the frequency response. The response is improved as we add more poles to the Transfer Function of the system, but then the order of the polynomial grows as we add more poles, making the system more complex to design. One way to simplify the complexity is to cascade blocks of simple Transfer functions. These blocks are made up of the first and second order systems of single real or single complex poles and zeros. Complex filters are implemented as cascades of the basic building blocks.

The following Transfer Functions provide the four basic types of filters, namely the low-pass, high-pass, band-pass and band-stop. The equivalent circuit matching the Transfer Function is being described in the next section of analog filters and the equivalent difference equations for the digital filters are being discussed in the next chapter.

First order low-pass filter Transfer Function

The Transfer Function of Equation 6.1 is a low-pass single pole filter with the cutoff frequency of p_1 and the gain of K at 0 frequency.

$$\frac{V_O}{V_I} = \frac{K}{1 + s/p_1} \tag{6.1}$$

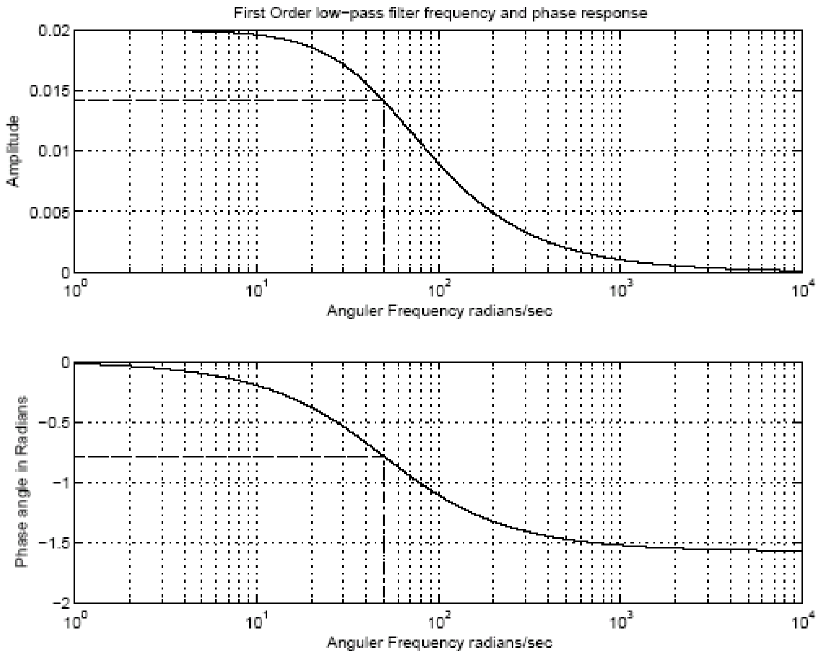
The magnitude response is

$$|H(j\omega)| = \left|\frac{K}{1 + j(\omega/p_1)}\right| = \frac{K}{\sqrt{1 + (\omega/p_1)}}$$

The phase function from the Equation 6.1 is

$$\arg(K) - \tan^{-1}\frac{\omega}{p_1} = -\tan^{-1}\frac{\omega}{p_1}$$

The normalized form can be derived with a known value of p_1 as $\omega_n = \dfrac{\omega}{p_1}$ and the normalized frequency and phase response of the low-pass transfer function is shown in the Figure 6.3.a. and 6.3.b



******Insert Figure 6.3 here*******
Figure 6.3. First order low-pass filer response, a) Magnitude response, b) Phase response

Second order low-pass filter Transfer Function

A 40 dB roll-off rate may be achieved with the addition of an extra pole resulting in the Transfer Function of Equation 6.2,

$$H(s) = \frac{V_O}{V_I} = \frac{K_1}{s^2 + \alpha s + \beta} \tag{6.2}$$

The magnitude response may be obtained by substituting by $s = j\omega$ in Equation 6.2

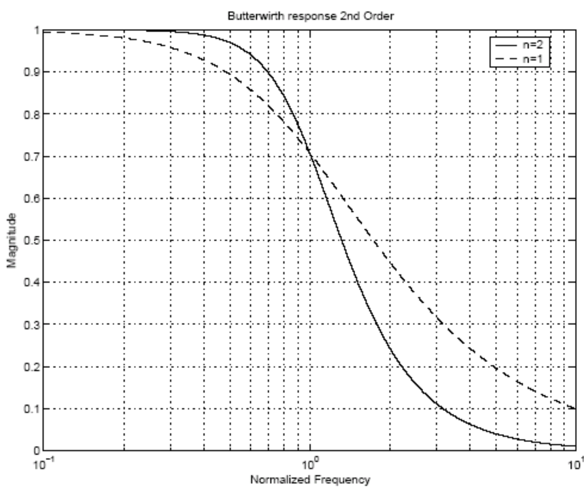
$$|H(j\omega)| = \left| \frac{K}{(\alpha - \omega^2) + (j\omega\alpha)} \right| = \frac{K}{\sqrt{(\alpha - \omega^2)^2 - (\omega\alpha)^2}}$$

The phase function from the Equation 6.2 is

$$\tan^{-1} \frac{j\alpha\omega}{(\alpha - \omega^2)}$$

The normalized form can be derived with a known value of β as $\omega_n = \dfrac{\omega}{\beta}$ and the normalized frequency and phase response of the low-pass transfer function is shown in the Figure 6.4.a. and 6.4.b

Note: When $\alpha = \sqrt{2}\beta$ the Equation 6.2 becomes a second order Butterworth polynomial that we will discuss later in the chapter.



******Insert Figure 6.4 here*******

Figure 6.4. Second order low-pass filer response, a) Magnitude response, b) Phase response

First order high-pass filter Transfer Function:

The following Transfer Function is for a high-pass single pole filter with the cutoff frequency of p_1 and the gain of K at 0 frequency.

$$H(s) = \frac{V_O}{V_I} = \frac{K}{1 + 1/sp_1} \tag{6.3}$$

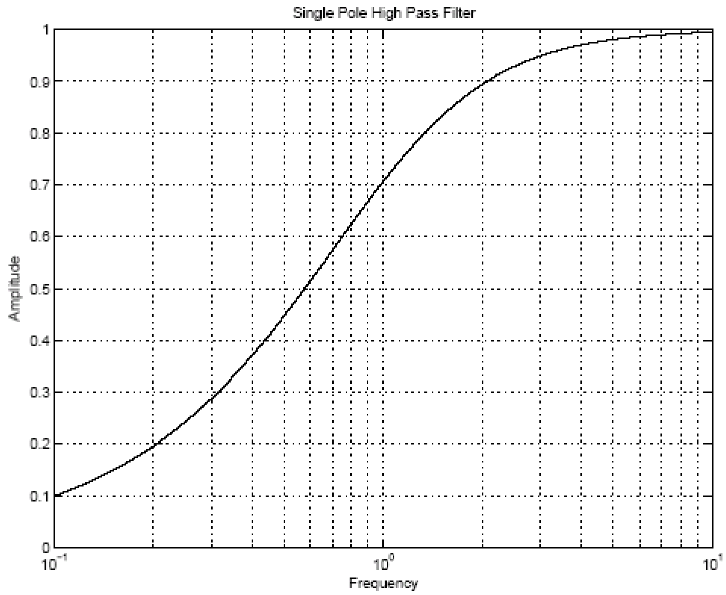
The magnitude response is

$$|H(j\omega)| = \left|\frac{K}{1 - j(1/\omega p_1)}\right| = \frac{K}{\sqrt{1 + (1/\omega p_1)}}$$

The phase function from the Equation 6.3 is

$$\arg(K) + \tan^{-1}\frac{1}{\omega p_1} = \tan^{-1}\frac{1}{\omega p_1}$$

The magnitude and phase response is shown in Figure 6.5.a. and 6.5.b



******Insert Figure 6.5 here*******

Figure 6.5 First order high-pass filer response, a) Magnitude response, b) Phase response

The second order high-pass filter Transfer Function

The second order gain of a high-pass filter is given as,

$$H(s) = \frac{V_O}{V_I} = \frac{K_3 s^2}{s^2 + b_1 s + b_0} \tag{6.4}$$

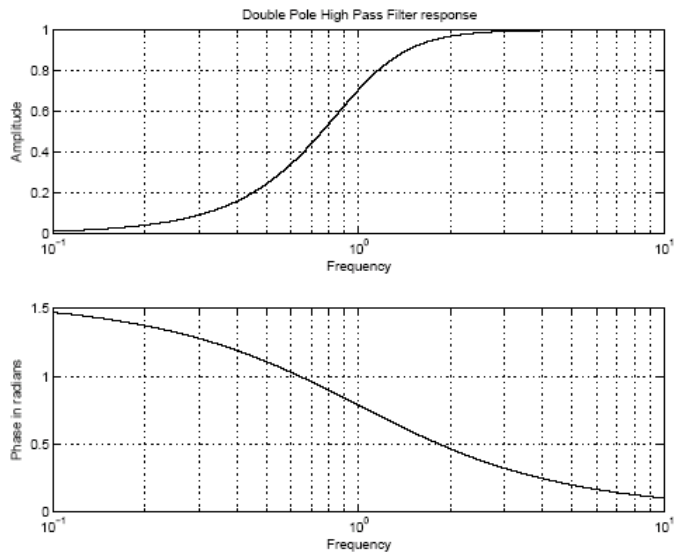
The magnitude response is

$$|H(j\omega)| = \left|\frac{-K_3\omega^2}{(b_0 - \omega^2) + (j\omega b_1)}\right| = \frac{K_3\omega^2}{\sqrt{(b_0 - \omega^2)^2 - (\omega b_1)^2}}$$

The phase function from the Equation 6.4 is

$$\tan^{-1}\frac{j\alpha\omega}{(\alpha - \omega^2)}$$

The normalized form can be derived with a known value of b_0 as $\omega_n = \dfrac{\omega}{b_0}$ and the normalized frequency and phase response of the second order low-pass transfer function is shown in the Figure 6.6.a. and 6.6.b



******Insert Figure 6.6 here*******

Figure 6.6 Second order highs filer response, a) Magnitude response, b) Phase response

The second order band-pass filter Transfer Function

The band-pass filter is usually defined in terms of the center frequency ω_0 and its two side frequencies ω_{3dB} where the gain falls off to the -3dB of the center frequency. The Transfer Function is given as

$$H(s) = \frac{V_O}{V_I} = \frac{K_3 s}{s^2 + (\omega_0/Q)s + \omega_0^2} \qquad (6.5)$$

Where $Q = (\omega_0/\omega_{3db})$

The width of the bandwidth is controlled by the quantity *Q*. If a narrow bandwidth is desirable then set the *Q* to a high value such as 10 or more.

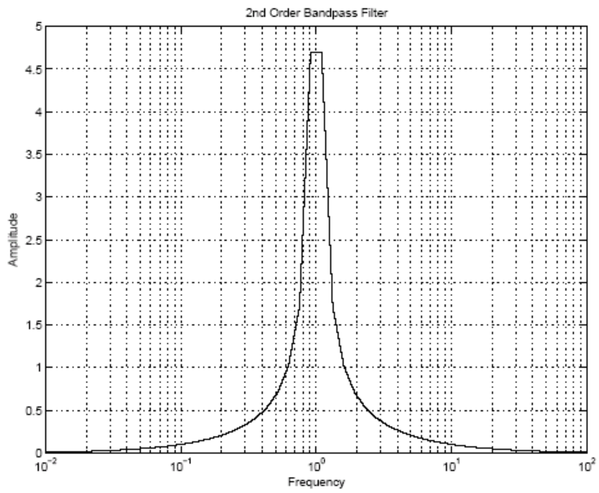
The magnitude response is

$$|H(j\omega)| = \left| \frac{K_3\omega}{(\omega_0^2 - \omega^2) + j(\omega_0/Q)} \right| = \frac{K_3\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + j(\omega_0/Q)^2}}$$

The phase function from the Equation 6.5 is

$$\tan^{-1}(j\omega) - \tan^{-1}\frac{j(\omega_0/Q)}{(\omega_0^2 - \omega^2)}$$

The magnitude and phase response is shown in Figure 6.7.a. and 6.7.b



******Insert Figure 6.7 here*******

Figure 6.7 Second order band-pass filer response, a) Magnitude response, b) Phase response

The second order band-stop filter Transfer Function

The band-stop filter is also known as notch filter. Similar to the band-pass filter, the Transfer Function is defined in terms of the center frequency ω_c that would be removed from the system and the two side frequencies ω_0 with the –3dB gain. The Transfer Function is given as

$$H(s) = \frac{V_O}{V_I} = \frac{K_4(s^2 + \omega_c)}{s^2 + (\omega_0/Q)s + \omega_0^2} \tag{6.6}$$

Where $Q = (\omega_0/\omega_{3db})$

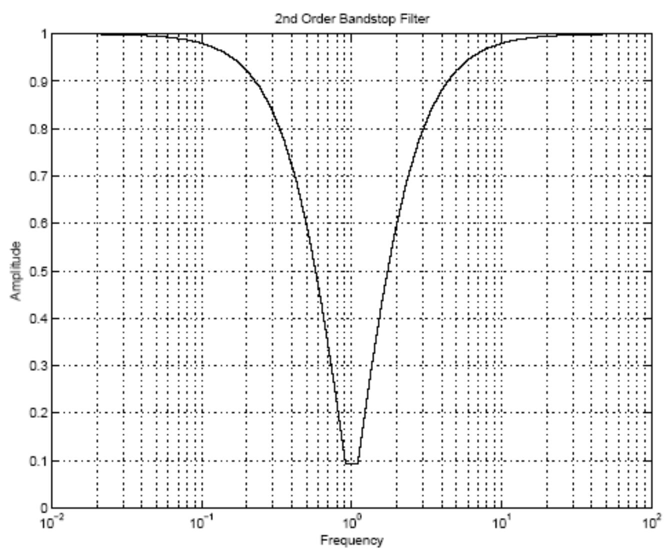
The magnitude response is

$$|H(j\omega)| = \left| \frac{K_4\sqrt{(\omega_c^2+\omega^2)^2}}{(\omega_0^2-\omega^2)+j(\omega_0/Q)} \right| = \frac{K_4(\omega_c^2+\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+j(\omega_0/Q)^2}}$$

The phase function from the Equation 6.6 is

$$\tan^{-1}(\omega_c^2+\omega^2)-\tan^{-1}\frac{j(\omega_0/Q)}{(\omega_0^2-\omega^2)}$$

The magnitude and phase response is shown in Figure 6.8.a. and 6.8.b



******Insert Figure 6.8 here*******

Figure 6.8 Second order band-pass filer response, a) Magnitude response, b) Phase response

Systems often require more sophisticated filter response then the 3dB gain and 20dB per decade drop in the roll-off rate as provided by the first order Transfer Function polynomials. There are different methods that enhance different aspects of the filter characteristics. The Chebychev filters improve upon the roll-off rate but add ripples in the pass-band region, whereas, the Butterworth filters give flatter response in the pass-band region but are not very efficient in the roll-off rate. We will discuss the Butterworth filters in the next section and the Chebyshev filter in the coming chapter.

Butterworth Filters

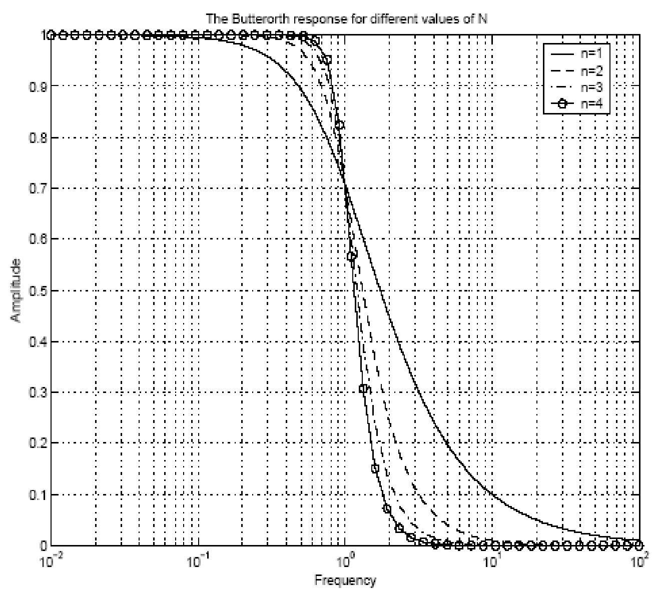
The goal in any filter design is to get as closer as possible to the ideal filter response. We would like to see a flatter response in the pass-band region and a steeper roll-off in the transition band. The Butterworth polynomials offer such a response. The response function is given as,

$$A(s) = \frac{1}{1+s^n}$$

And the magnitude squared is,

$$|A(\omega)|^2 = \frac{1}{1+(j\omega)^{2n}} \tag{6.7}$$

If ω is normalized with respect to the cutoff frequency $\omega_o = \frac{\omega}{\omega_{Cutoff}} = 1$, the magnitude response of Equation 6.7 becomes 0.707 for all values of *n.* The contribution becomes less significant for the $\omega_o < 1$ and for $\omega_o > 1$ the magnitude approaches 0 faster with increasing *n*. The result is a flatter response for frequencies less than the cutoff frequency and steeper roll for the frequencies greater than the cutoff frequency with increasing *n* as shown in the Figure 6.9.



******Insert Figure 6.9 here*******
Figure 6.8 The Butterworth response with different values of *n*.

We can approximate the Equation 6.7 using $1-\omega \approx 1/1+\omega ... for \omega << 1$, resulting in the Equation 6.8.

$$|H(j\omega)|^{2N} = \frac{1}{1+(1/\omega_C)^2\omega^2+(1/\omega_C)^4\omega^4\cdots(1/\omega_C)^{2N}\omega^{2N}} \quad (6.8)$$

Let's analyze the Transfer Function of the simple low-pass filter of the Equation 6.7, rewritten as the square magnitude in the following Equation.

$$|H(j\omega)|^2 = \frac{1}{1+(\omega/\omega_C)^2}$$

Where ω_C is the cutoff frequency and as $\omega \to \infty$ the response approaches 0 and when $\omega = 0$ the response is 1. In between there is a gradual loss of magnitude. Precisely, at cutoff frequency of ω_C the magnitude is –3dB. This is equivalent of the first order Butterworth response as shown in Equation 6.7. The expansion of coefficients $(\omega_0)^N$ may be explained as follows,

Butterworth expansion

If the power vectors of the pole frequency ω_C were to be drawn on a circle, (with the radius (ω_C) that can be normalized and set equal to 1, all vectors would appear as equally spaced on a semi circle. Since, multiplying a unit vector to itself simply shifts the vector to an angle, the roots of Equation 6.7 are all shifted by an angle π/N with respect to the real axis. This could be evident from the following analogy,

The poles are actually roots of the equation,

$$(e^{j\pi})^N = -1 \quad (6.9)$$

The Figure 6.10 indicates pole locations for various values of *N*. Except for the first pole, all others are complex conjugates. If *N* is even, all poles are complex conjugates and when *N* is odd, there is only one real and the rest are complex conjugates poles. An nth order Butterworth Filter is defined as,

$$|H(s)|^2 = \frac{1}{\prod_{i=1}^{N}(s-s_i)} = \frac{1}{(s-s_1)(s-s_2)\cdots(s-s_N)}$$

Where $s_i = e^{j\pi\left[2i+n-1/2n\right]} = \cos(\pi\frac{2i+n-1}{2n}) + j\sin(\pi\frac{2i+n-1}{2n})$

The following calculations indicate the pole vectors for different values of *N*,

$N = 1 \qquad s_1 = \cos(\pi) + j\sin(\pi) = 1$

$$H(s) = \frac{1}{(s+1)}$$

$N = 2$

$$s_1 = \cos(\pi 3/4) + j\sin(\pi 3/4) = -.707 + j.707$$
$$s_2 = \cos(\pi 5/4) + j\sin(\pi 5/4) = -.707 - j.707$$
$$H(s) = \frac{1}{(s_2^2 + \sqrt{2}s_2 + 1)}$$

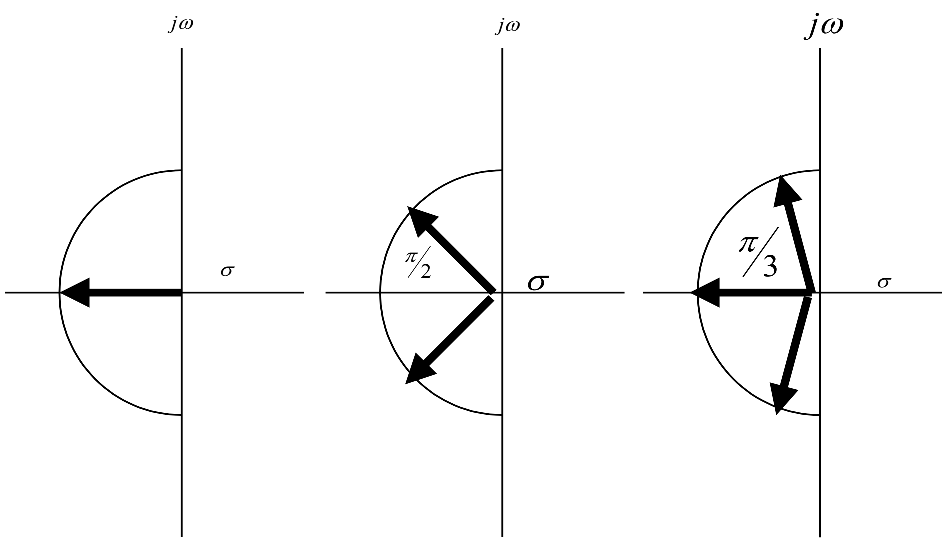
$N = 3$

$$s_1 = \cos(\pi 2/3) + j\sin(\pi 2/3) = -.5 + j.866$$
$$s_2 = \cos(\pi) + j\sin(\pi) = 1$$
$$s_3 = \cos(\pi 2/3) + j\sin(\pi 2/3) = -.5 - j.866$$
$$H(s) = \frac{1}{(s_3^3 + 2s_3^2 + s_3 + 1)} = \frac{1}{(s_3 + 1)(2s_3^2 + s_3 + 1)}$$

$N = 4$

$$s_1 = \cos(\pi 5/8) + j\sin(\pi 5/8) = -.382 + j.924$$
$$s_2 = \cos(\pi 7/8) + j\sin(\pi 7/8) = -.924 + j.383$$
$$s_3 = \cos(\pi 9/8) + j\sin(\pi 9/8) = -.924 - j.383$$
$$s_4 = \cos(\pi 11/8) + j\sin(\pi 11/8) = -.382 - j.924$$

$$H(s) = \frac{1}{(s_4^4 + 2.61s_4^3 + 2.61s_4^2 + s_4 + 1)} = \frac{1}{(s_4^2 + 0.765s_4 + 1)(s_4^2 + 0.765s_4 + 1)}$$



****Insert Figure 6.10 here*****
Figure 6.10 Pole locations of Butterworth response

Next, we discuss the implementation details of the Transfer Functions in terms of the electronic circuit designs for the analog filters (the digital filters are being discussed in the next chapter).

Analog Filters

In analog domain a filter is a circuit that produces the expected response specified by the corresponding filter Transfer Function. We have seen in the previous chapter of the Solutions of Differential Equations, how a first and second order differential equation's response acts as a frequency selector. The higher frequencies are suppressed from the voltage across the capacitor, in a network of resistor and capacitor. Similarly, the lower frequencies are suppressed from the voltage across the resistor, in the network of a resistor and capacitor.

The filter circuitry in our designs will have basically two types of components, the passive components (the resistors and capacitors) and the active components (the op-amps), hence, the name ***Active RC Filters***. The op-amps play dual role in the analog filters. They not only simulate inductors (if needed as a second order section) but also act as non-interacting blocks to prevent loading effects on the passive components. Since, op-amps play important role in setting up the network equations, a brief overview is presented next as a refresher.

Op-amp as differential amplifier

A differential amplifier's function is to amplify the difference between two signals. The basic schematic of the op-amp is represented in the Figure 6.11.a and the corresponding circuit model is shown in Figure 6.11.b The R_i is the differential input impedance of an extremely high value with practically no current flowing through it. The R_o is the output impedance, a negligible quantity as if it does not exist in the circuit. The amplifier gain of Figure 6.11.a is expressed as

$$v_o = a_o(v_1 - v_2)$$

Where a_0 is the open loop gain of the op-amp when there is no feed back and is usually very large in the range of 10^5 to 10^6, considered infinite for an ideal op-amp. We will use the equivalent of voltage controlled voltage source as shown in the Figure 6.11.b in the loop analysis of network when we use op-amps in our filter design.

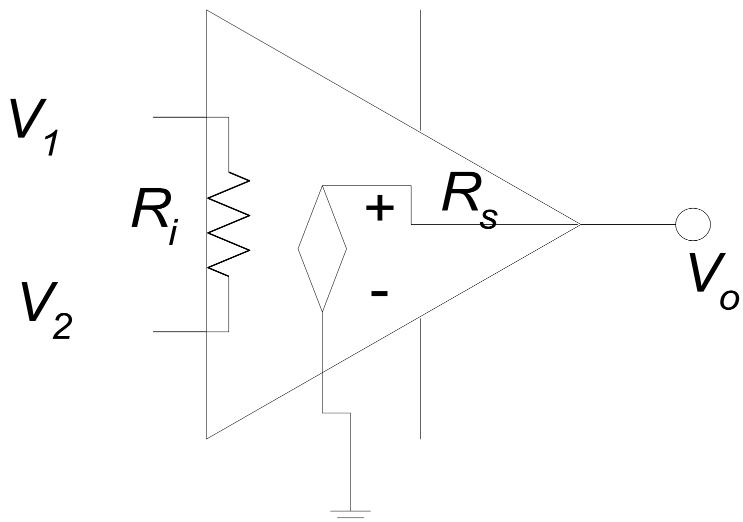
Two things need to be remembered when dealing with op-amps,

a) Infinite input resistance means the current into the inverting input is zero:

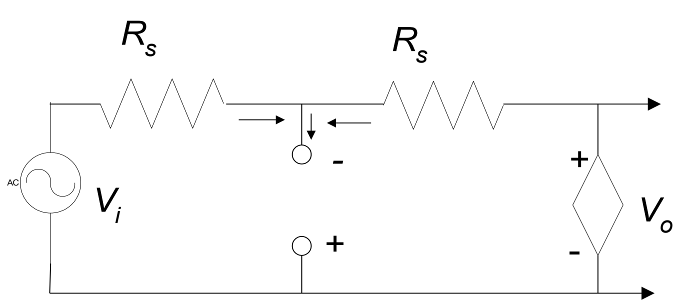
$i_- = 0$

b) Infinite gain means the difference between v_+ and v_- is zero:

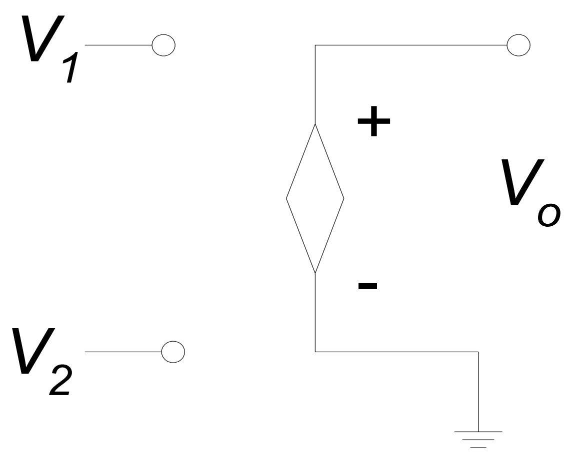
$v_+ - v_- = 0$



******Insert Figure 6.11a here*******



******Insert Figure 6.11b here*******



****Insert Figure 6.11c here*****
Figure 6.11 a) Op-amp as a differential amplifier, b) Equivalent circuit model
c) Input and output voltage across the op amp.

The two most common configurations are provided next to help identify the loop equation created by a filter circuit.

Inverting Amplifier

The Figure 6.12.a shows an inverting amplifier configuration, the Figure 6.12b shows the summation of current at the inverting input.

The current passing through the two resistors is equal,

$$i_1 = i_2 = i$$

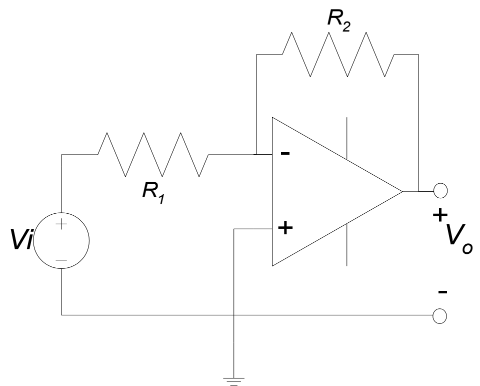
Since all the current is passing through the source resistor R_1 and the feedback resistor R_2, we have the following voltage drop

$v_i = R_1 i$ and $v_o = -R_2 i$

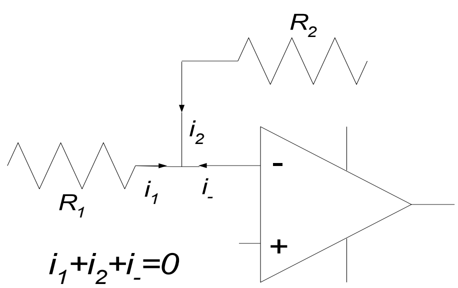
Resulting in the amplifier voltage gain of

$$\frac{v_o}{v_i} = -\frac{R_2}{R_1}$$

The op amp will provide whatever output voltage is necessary so that both the input voltages equal.



****Insert Figure 6.12a here*****



****Insert Figure 6.12b here*****

Figure 6.12. a) The inverting configuration of an op-amp b) Summation of the current at the node

Non-Inverting Amplifier

The Figure 6.13 is a non-inverting amplifier configuration showing the summation of the current at the inverted input.

Since no current is passing through any of the op-amp input, the two input voltages are equal,

$$v_+ = v_-$$

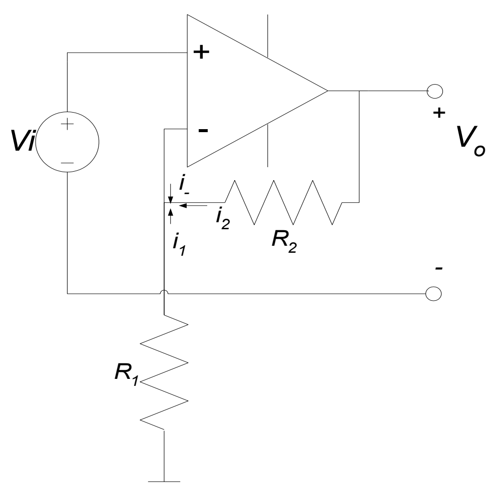
The voltage across $v_- = \dfrac{R_2}{R_1 + R_2} v_o$ and $v_+ = v_- = v_i$

The voltage gain is

$$\frac{v_o}{v_i} = \frac{R_1 + R_2}{R_1} = 1 + \frac{R_2}{R_1}$$

Resulting in the voltage gain of

$$v_o = \frac{R_1 + R_2}{R_1} v_i$$



******Insert Figure 6.13 here*******

Figure 6.13. The non-inverter configuration of an op-amp

With the basic configurations of op-amps in place, we are ready to design some simple building block analog filters using the resistors, capacitors and op-amps.

Active *RC* Filters

The design of the analog filters requires establishing the loop equations of the four basic types of the filter Transfer Functions, namely, the low-pass, high-pass, band-pass and band-stop filters. The requirements are specified in terms of the cutoff frequencies, establishing the poles and zeros of the filter's Transfer Function and improvement is achieved by cascading several blocks in series. The network loop equation parameters directly correspond to the component values and the design requirement is simply a matter of choice between the different component compositions.

Single pole low-pass filter

The circuit of Figure 6.14 is a single pole low pass filter, corresponding to the Transfer Function of Equation 6.10.The output V_c measured on the capacitor C is given as,

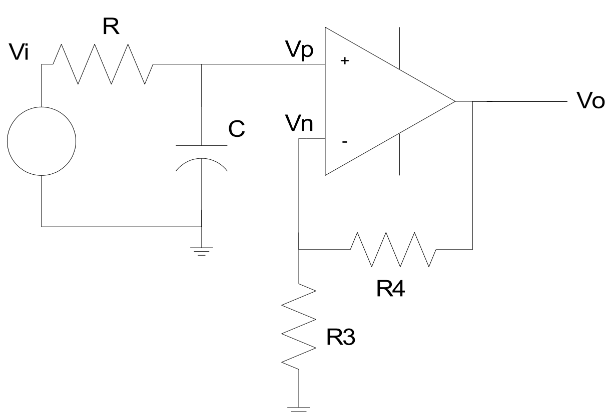
$$H(s) = \frac{V_O}{V_I} = \frac{Z_C}{Z_R + Z_C}$$

Substituting for the value of $Z_C = {}^{1}\!/_{j\omega C}$ and $Z_R = R$,

$$H(j\omega) = \frac{V_O}{V_I} = \frac{1}{1 + j\omega RC}$$

Presenting the Transfer Function in a vector form,

(6.10)



******Insert Figure 6.14 here*******
Figure 6.14. First order low-pass active RC filter

Design considerations

From the Equation 6.10 we can draw the following filter characteristics,

The cutoff frequency $\omega_c = {}^{1}\!/_{RC}$ (6.11)

The frequency response magnitude,

$$| H(j\omega) |= \frac{1}{RC} \times \frac{1}{\sqrt{({}^{1}\!/_{RC})^2 + \omega^2}} \tag{6.12}$$

The Gain in decibels

$$20\log | H(j\omega) |= 20\log | \frac{1}{RC} | -20\log | \sqrt{({}^{1}\!/_{RC})^2 + \omega^2} | \tag{6.13}$$

The phase change

$$\Phi = -\arctan(\omega RC)$$

A plot of the frequency response magnitude and phase change is presented in Figure 6.5

Example 6.1

Design a low-pass filter with the cutoff frequency of $\omega = 2\pi 1000 rad / s^{-1}$.

Solution:

From the Equation 6.10 we have,

$$1/RC = 2\pi 1000 rad / s^{-1}$$

Since we have two element values to choose from, by selecting R=10 K, we get

$$C = 1/ 2\pi 1000 * 10000 = 1.57 \times 10^{-7} \mu F$$

Example 6.2

Define the cutoff frequency for R=10 K and C=0.001 in the Figure 6.14

Solution:

From the Equation 6.11,

$$\omega_c = 1 \Big/ (10 \times 10^3 \times 0.001 \times 10^{-6}) = 100 krad/\sec$$

Example 6.3

Define the magnitude in dB at frequency 10 KHz, for *R*=1K and $C = 0.01\mu F$ in the Figure 6.1

Solution:

$$\omega = 2\pi f = 2\pi \times 10^4 \, rad/s$$

From the Equation 6.11,

$$1/RC = 1/10^{-8} \times 10^3 = 10^5 \, rad/\sec$$

From the Equation 6.12,

$$|H(j\omega)| = \frac{10^5}{\sqrt{(2\pi \times 10^4)^2 + (10^5)^2}} = 0.97$$

From the Equation 6.13,

Magnitude in dB $\quad 20\log(0.97) = -0.26 dB$

Second order low-Pass filter

The roll-off rate could be improved to 40 dB with the implementation of the circuit in Figure 6.15 corresponding to the Transfer Function of Equation 6.14.

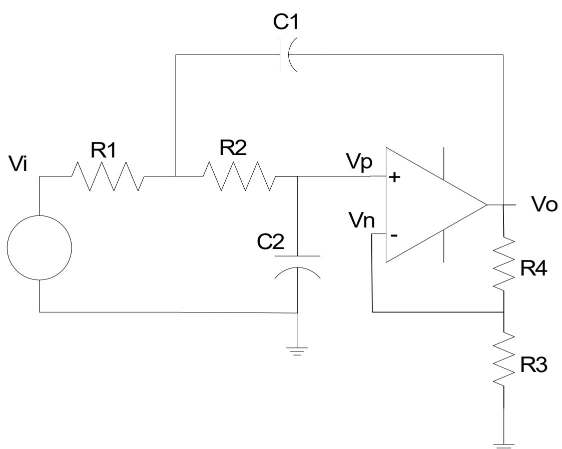
$$H(s) = \frac{H_1}{\omega^2 + \alpha\omega + \omega_C^2} \tag{6.14}$$

For a second order Butterworth response

$$\alpha = \sqrt{2}\omega_C$$

The loop equation is given as

$$H(s) = \frac{V_o}{V_I} = \frac{K/R_1C_1\,R_2C_2}{s^2 + s(1/R_1C_1 + 1/R_2C_1 + 1/R_2C_2 - K/R_2C_2) + 1/R_1C_1\,R_2C_2} \qquad (6.15)$$



******Insert Figure 6.15 here*******

Figure 6.15. The frequency and phase response of a second order low-pass active RC filter

Source: Operational Amplifiers and Linear Integrated Circuits.
Robert F. Coughlin, Frederick F. Drisoll. Prentice Hall Publisher

Example 6.4

Design a Butterworth Filter with a -3dB gain at $2\pi 10000 rad/\sec$ and a gain of 10 at 0 dB. The circuit of Figure has input resistor R_1=1K.

Solution:

Equating the Transfer Function of Equation 6.14 and the loop Equation 6.15, we get three Equations and 5 unknowns R1, R2, C_1. C2 and K,

$$H(0) = 10 = K/R_1C_1R_2C_2 \qquad (6.16)$$

$$\sqrt{2}\omega_C = (1/R_1C_1 + 1/R_2C_1 + 1/R_2C_2 - K/R_2C_2) \qquad (6.17)$$

$$\omega_C^2 = 1/R_1C_1R_2C_2 \qquad (6.18)$$

Let's use normalized values of R1=1, C1=1, $\omega_C = 1$ and solve for the other unknowns using Equations 6.16, 6.17 and 6.18.

$normalized R_2 = 0.106$ $\quad$ $normalized C_2 = 9.414$ $\quad$ $K = 10$

The actual values are

$R_1 = 1K$ $\quad$ $R_2 = 1K \times 0.106 = 106\Omega$ $\quad$ $C_1 = .15\mu F$ $\quad$ $C_2 = .016\mu F$

The band-pass filter

All resonant circuits act as band-pass filters. The circuit of Figure 6.16 corresponds to the Transfer Function of the Equation 6.19 that allows a certain band of frequencies to pass through while suppressing the rest.

$$H(s) = \frac{H_2 s}{s^2 + (\omega_0/Q)\omega_0 + \omega_0^2} \tag{6.19}$$

There is one frequency (the resonant frequency also called the center frequency) that has the maximum output voltage V_{max}. The frequency to the left of the center frequency with the magnitude 0.707 V_{max} is the low cutoff frequency and the frequency to the right with $0.707 V_{max}$ is the high cutoff frequency. The bandwidth is between the low and high cutoff points.

The loop equation of the circuit of Figure 6.16 is given as

$$H(s) = \frac{V_o}{V_I} = \frac{-(s/C_1R_1)}{s^2 + s(C_1 + C_2)/C_1C_2R_2 + 1/C_1R_1C_2R_2} \tag{6.20}$$

Equating the parameters of the Equation 6.19 and 6.20

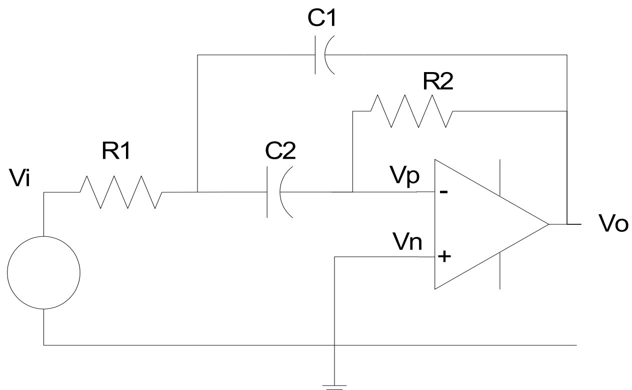
$H_2 = 1/C_1R_1$

The cutoff frequency

$$\omega_0^2 = 1/C_1R_1C_2R_2 \tag{6.21}$$

The bandwidth quality factor

$$(\omega_0/Q) = (C_1 + C_2)/C_1C_2R_2 \tag{6.22}$$



******Insert Figure 6.16 here*******
Figure 6.16. The frequency and phase response of a second order band-pass active RC filter

Example 6.5:

Design a band-pass filter with Q = 2 and $\omega = 2\pi 1000 rad/s^{-1}$.

Solution:

There are four independent values to choose from R_1, R_2, C_1 and C_2. The choice could be narrowed down by making $C_1=C_2=C$ *and* $R_1=R_2=R$ and substituting the values into Equation 6.21 and 6.22

$$\omega_0^2 = 2/(CR)^2 = [2\pi 1000]^2 \tag{6.23}$$

$$(\omega_0/Q) = (1/CR) = 2\pi 1000 \tag{6.24}$$

Let R = 10K and From Equation 6.23 and 6.24 we get

$C1 = C2 = 1.6 \times 10^{-6}\,\mu F$ $\qquad R_1 = R_2 = 10K$

Single pole high-pass filter

Implementing the Transfer Function of Equation 6.1 gives a single pole high-pass filter. Notice, swapping the resistor and capacitor of low-pass filter of Figure 6.14 converts it into a high pass filter as shown in Figure 6.17.

The loop Equation is described as,

$$H(j\omega) = \frac{V_O}{V_I} = \frac{1}{1 - j(1/\omega RC)} \tag{6.25}$$

Design considerations

From the Equation 6.25 we can draw the following filter characteristics,

The cutoff frequency $\omega_c = 1/RC$ (6.26)

The frequency response magnitude,

$$|H(j\omega)| = \frac{1}{RC} \times \frac{1}{\sqrt{(1/RC)^2 + \omega^2}} \tag{6.27}$$

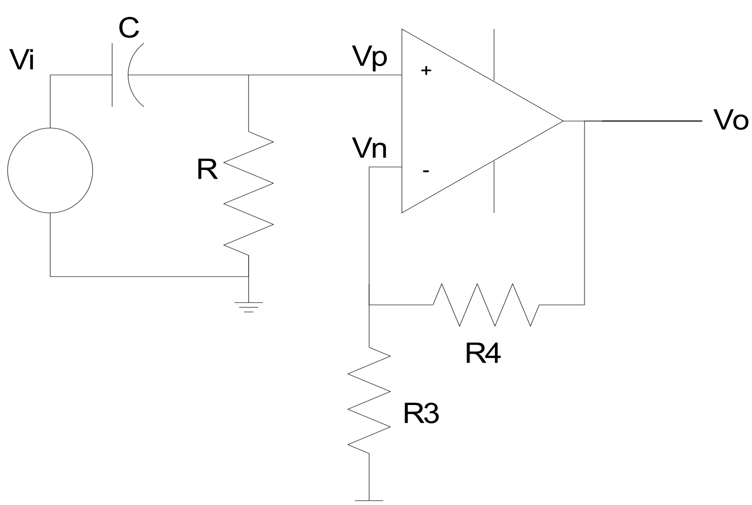
The Gain in decibels

$$20\log|H(j\omega)| = 20\log\left|\frac{1}{RC}\right| - 20\log\left|\sqrt{(1/RC)^2 + \omega^2}\right| \tag{6.28}$$

The phase change

$$\Phi = -\arctan(1/\omega RC) \tag{6.29}$$

A plot of the frequency response magnitude and phase change is presented in Figure 6.5



****Insert Figure 6.17 here*****
Figure 6.17. The frequency and phase response of a first order high-pass active RC filter

Source: Operational Amplifiers and Linear Integrated Circuits.
Robert F. Coughlin, Frederick F. Drisoll. Prentice Hall Publisher

Example 6.6

Design a high-pass filter with cutoff frequency of $\omega = 2\pi 1000 rad/s^{-1}$ and the transition band of 20 dB per decade.

Solution:

The single pole solution of Equation 6.25 satisfies 20 dB requirements.

Substituting the cutoff frequency $\omega = 2\pi 1000 rad/s^{-1}$ into Equation 6.26, we get

$$(1/RC) = 2\pi 1000$$

If we pick R=1K the capacitor $C = 1.57 \times 10^{-6}\ \mu F$

Example 6.7

Define the magnitude and phase of the filter output of the Example 6.6 at 100 Hz.

The magnitude from the Equation 6.27

$$|H(j\omega)| = 2\pi 1000 \times \frac{1}{\sqrt{(2\pi 1000)^2 + 100^2}} = 0.999$$

The log magnitude from the Equation 6.28

$$20\log|0.999| = -0.01$$

The phase from the Equation 6.28

$$\Phi = -\arctan(2\pi 100 \times 2\pi 1000) = 0.001\deg$$

Second order High-Pass filter

Implementing the Transfer Function of Equation 6.30 provides a 40 dB roll-off rate.

$$H(s) = \frac{H_2\omega}{\omega^2 + b_1\omega + b_0} \tag{6.30}$$

The circuit of Figure 6.18 has the loop Equation matching the Transfer Function

$$H(s) = \frac{V_o}{V_I} = \frac{Ks^2}{s^2 + s(1/R_1C_1 - K/R_2C_1 + 1/R_2C_2 + K/R_2C_2) + 1/R_1C_1\, R_2C_2} \qquad (6.31)$$

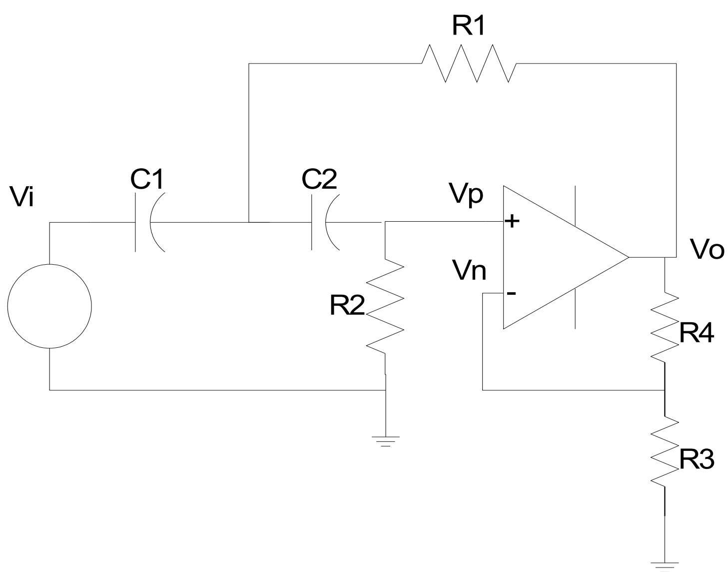
There are five unknowns R1, R2, C1, C2 and the gain K with three equations,

The Gain $\quad H = K$

The cutoff frequency $\omega_c^2 = b_o = \dfrac{1}{R_1R_2C_1C_2}$

For a second order Butterworth response

$$b_1 = \sqrt{2}\omega_C = \frac{1-K}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C1}$$



******Insert Figure 6.18 here*******
Figure 6.18. The frequency and phase response of a second order high-pass active RC filter

Source: Operational Amplifiers and Linear Integrated Circuits.
Robert F. Coughlin, Frederick F. Drisoll. Prentice Hall Publisher

Example 6.8:

Design a Butterworth Filter with a -3dB gain at $2\pi 10000 rad/\sec$ and a gain of 10 at $2\pi 10000 rad/\sec$ dB. The circuit of Figure has input resistor R_1=1K.

Solution:

Equating the Transfer Function of Equation 6.1 and the loop Equation 6.1, we get three Equations and 5 unknowns R1, R2, C_1. C2 and K. We could normalized values of R_1=1,C_1=1, $\omega_o = 1$ and obtain the three unknown values in normalized form,

$K = 10$, $C_2 = 9.414$ $R_2 = 1/9.414$ $R_1 = 1$ $C_1 = 1$

The actual values are

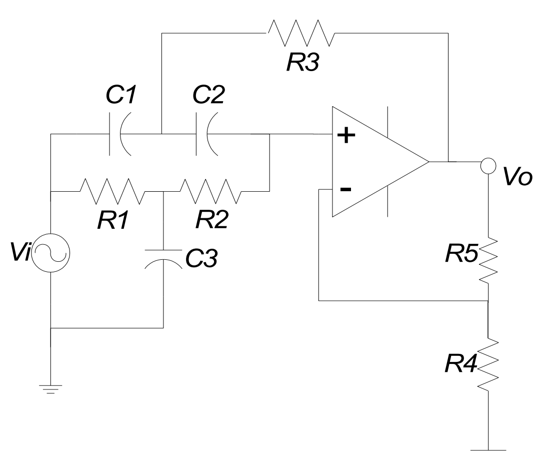
$R_1 = 1K$ $R_2 = 1K \times 0.11 = 111\Omega$ $C_1 = .016\mu F$ $C_2 = .15\mu F$

Band-stop filter

A band stop response is achieved by canceling the band-pass response with a pure resonant response from the numerator polynomial.

The circuit shown in the Figure 6.19 has a gain,

$$H(s) = \frac{H_4(s^2 + \omega^2)}{s^2 + (\omega_p / Q)\omega_0 + \omega_p^2} \qquad 6.32)$$



******Insert Figure 6.19 here*******
Figure 6.19. Circuit for a second order band-stop active RC filter

The loop equations are given as

The loop equations are given as

$$H(s) = \frac{V_o}{V_I} = \frac{K(C_1 C_2 s^2 + 1/R_1 R_2}{C_1 C_2 s^2 + s\left[(C_1 + C_2)/R_2 + C_2(1/R_1 + 1/R_2)(1-K)\right] + 1/R_1 R_2}$$

There are five design parameters, R1,R2, C1, C2 and the gain K. and three Equations, making C1=C2=C and R1=R2=R we can reduce the number of variables to three, and solve for them using,

$$H(s) = \frac{V_o}{V_I} = \frac{K(C^2 s^2 + 1/R^2)}{C^2 s^2 + 2Cs/R + 1/R^2}$$

Conclusion

The goal in this chapter was to introduce the Transfer Functions of different filter types including, the low-pass, high-pass, band-pass and band-stop filters and design the analog circuits that match the corresponding Transfer Function of each filter types.