Chapter5 - Laplace Transform and z Transform

In this chapter

- *Laplace Transform*
- Transfer Function
- The z-Transform

Introduction

We discussed in chapter 4 the method of Convolution as a solution of differential equations. The problem with Convolution is that it is a specific solution and not a statement of system's behavior. It cannot be used as a model for the system. We need an expression that could describe the outcome of a system in terms of its dependent and independent variables, so design possibilities could be exploited. A plot of a frequency response is the most desirable picture in a digital or analog filter design and for that an algebraic form of a differential equation is required. We need to know what the output amplitude and phase change would be, given a set of input frequency. The amplitude tells us how much gain or loss is impeded in the output and the phase tells us the fraction of a complete cycle elapsed from a specified reference point. In this chapter, we would solve these problems with the help of the Laplace Transform.

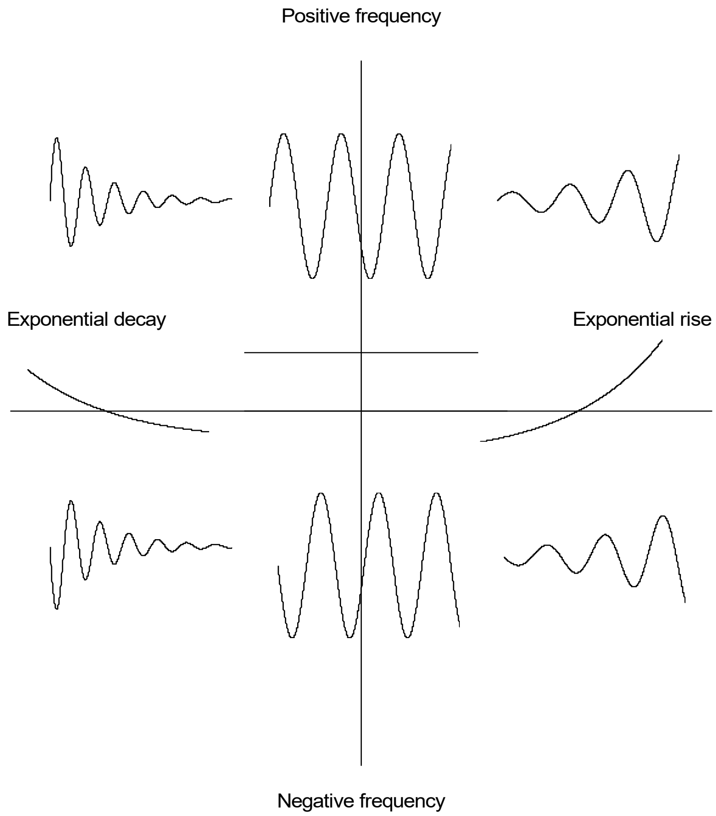
We discovered in the previous chapter that the exponentially decaying sinusoids, whether over-damped, under-damped or critically-damped are in-deed solutions of linear differential equations and the solution is only a matter of identifying the appropriate exponent coefficients. If the coefficients are real then the solution is simply an exponential decaying response, but if the coefficients are complex then we have a sinusoidal response with a damped frequency of oscillation. The damping factor determines the exponential decaying amplitude. Similar to the Fourier Transform that identifies the component frequencies in a system, the Laplace Transform identifies the exponential decaying frequencies in a system. Just like a logarithm converts a multiplication into an addition, the Laplace Transform converts a differential into a multiplication and an integral into a division operation. Another consequence of Laplace Transform is that the process eliminates the time dependency from the system, leaving behind only frequency as the independent variable.

The Laplace Transform

If you recall the theory of Fourier Transform being discussed in Chapter one, you would notice that we used the orthogonal property of sinusoidal function to identify the constituent frequencies of a given function. Broadening the concept and taking into consideration of the exponential damping factor, we introduce the following integral as the Laplace Transform of a function,

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt \tag{5.1}$$

The quantity $s = \sigma + j\omega$ is essentially a combination of exponentially decaying factor and a sinusoidal function of frequency ω. If σ is considered 0 (meaning, insignificant damping effect) then the remaining exponent is the Fourier integral in the form of Euler's identity $e^{-j\omega t}$. In reality, as time goes by, all exponential decay settle down to insignificant values. In essence, the Laplace integral is only filling in for the missing transient response from the Fourier integral. The Figure 5.1 is a map of the Laplace integral over the entire range of exponent decay σ and the frequency ω. The centerline is the Fourier integral, the left axis is the exponential decay and the right axis is the exponential rise. The result of the Laplace integral is a function of frequency spectrum that identifies component frequencies in the system.



****** Insert Figure 5.1 here ******
Figure 5.1 A map of Laplace integrals

It is customary to indicate the Transform in capital letters. The integral is definite and the range is from -0 to infinity, thus eliminating the time factor, leaving behind a function that depends upon the complex variable *s* only. The range of integration is taken from –0 to incorporate the impulse function that starts at *t=0*. Notice that if *s* is positive then the integral becomes infinite and that possibility needs to be avoided.

When we discuss the properties of the Laplace Transform, it would become obvious that it provides linearity to a system. Think of the Transform as an operator that when applied, transforms differentials and integrals into simpler algebraic operations. The followings are the properties of the Laplace Transform.

Linearity property of the Laplace Transform

The Transform of sum of two functions is equal to the sum of individual Transforms; the linearity is evident from the following proof,

$$L[f_1(t)+f_2(t)] = \int_{0-}^{\infty} e^{-st}[f_1(t)+f_2(t)]dt$$

$$L[f_1(t)+f_2(t)] = \int_{0-}^{\infty} e^{-st} f_1(t)dt + \int_{0-}^{\infty} e^{-st} f_2(t)dt$$

$$L[f_1(t)+f_2(t)] = L[f_1(t)] + L[f_2(t)]$$

Some useful Transformations
Let's derive Transformation of some basic functions,

Imulse function:

The impulse function has a non zero value only at *t=0* and at *t=0* the exponent is evaluated as 1, and by definition the integral of the impulse function is a unit area, thus the Laplace Transform of an impulse function is 1 as shown as shown in the Equation 5.2.

$$L[\delta(t)] = \int_{0-}^{\infty} \delta(t)e^{-st}dt = \int_{0-}^{\infty} \delta(t)dt = 1 \qquad (5.2)$$

Unit step function:

$$L[u(t)] = \int_{0-}^{\infty} u(t)e^{-st}dt = \int_{0-}^{\infty} e^{-st}dt = \left.\frac{e^{-st}}{s}\right|_{0-}^{\infty} = \frac{1}{s} \qquad (5.3)$$

The Equation 5.3 is valid for positive s only.

Unit step function with complex value s:

If s is a complex value of the form $s = \alpha + j\omega$ then the integral produces a complex quantity, as shown below,

$$L[u(t)] = \int_{0-}^{\infty} u(t)e^{-(\alpha+j\omega)t}dt = \int_{0-}^{\infty} e^{-(\alpha+j\omega)t}dt = \left.\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right|_{0-}^{\infty} = \frac{1}{\alpha+j\omega} \qquad (5.4)$$

Exponent function:

$$L[e^{at}] = \int_{0-}^{\infty} e^{at}e^{-st}dt = \int_{0-}^{\infty} e^{-(s-a)t}dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_{0-}^{\infty} = \frac{1}{s-a} \qquad (5.5)$$

Cosine function:

$$L[\cos(\omega t)] = \int_{0-}^{\infty} \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})e^{-st}\,dt = \frac{1}{2}\int_{0-}^{\infty} e^{-(s-j\omega)t}\,dt + \frac{1}{2}\int_{0-}^{\infty} e^{-(s-j\omega)t}\,dt$$

$$L[\cos(\omega t)] = \frac{1}{2}\frac{1}{s-j\omega} + \frac{1}{2}\frac{1}{s+j\omega} = \frac{s}{s^2+\omega^2} \qquad (5.6)$$

Sine function:

$$L[\sin(\omega t)] = \int_{0-}^{\infty} \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})e^{-st}\,dt = \frac{1}{2}\int_{0-}^{\infty} e^{-(s-j\omega)t}\,dt + \frac{1}{2}\int_{0-}^{\infty} e^{-(s-j\omega)t}\,dt$$

$$L[\sin(\omega t)] = \frac{1}{2j}\frac{1}{s-j\omega} + \frac{1}{2j}\frac{1}{s+j\omega} = \frac{\omega}{s^2+\omega^2} \qquad (5.7)$$

Exponentially decaying cosine function

$$L[e^{\alpha t}\cos(\omega t)] = L[e^{\alpha t}\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})e^{st}]$$

$$L[e^{\alpha t}\cos(\omega t)] = \frac{1}{2}\int_{0-}^{\infty} e^{(\alpha+j\omega)t}e^{-st} + \frac{1}{2}\int_{0-}^{\infty} e^{(\alpha-j\omega)t}e^{-st}$$

$$L[e^{\alpha t}\cos(\omega t)] = \frac{1}{2}\frac{1}{s-(\alpha+j\omega)} + \frac{1}{2}\frac{1}{s-(\alpha-j\omega)} = \frac{s-\alpha}{(s-\alpha)^2+\omega^2} \qquad (5.8)$$

Exponentially decaying sine function

$$L[e^{\alpha t}\sin(\omega t)] = L[e^{\alpha t}\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})e^{st}]$$

$$L[e^{\alpha t}\sin(\omega t)] = \frac{1}{2j}\int_{0-}^{\infty} e^{(\alpha+j\omega)t}e^{-st} - \frac{1}{2j}\int_{0-}^{\infty} e^{(\alpha-j\omega)t}e^{-st}$$

$$L[e^{\alpha t}\sin(\omega t)] = \frac{1}{2j}\frac{1}{s-(\alpha+j\omega)} - \frac{1}{2j}\frac{1}{s-(\alpha-j\omega)} = \frac{\omega}{(s-\alpha)^2+\omega^2} \qquad (5.9)$$

The Rules of Differentiation

The differential and integral rules are the two most important properties of Laplace Transform that translate a differential equation into a simple algebraic linear equation. We can accomplish the Transform of a derivative by using the method of integration by parts as shown in the following operation,

$$L[\frac{df(t)}{dt}] = \int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} = e^{-st} f(t) \Big|_{0-}^{\infty} + s \int_{0-}^{\infty} f(t) e^{-st}$$

$$L[\frac{df(t)}{dt}] = sL[f(t)] \tag{5.10}$$

$$L[\frac{d^2 f(t)}{dt^2}] = s^2 L[f(t)] \tag{5.11}$$

The Equation 5.10 and 5.11 simply state that the Transform of a derivative of a function is equal to the variable s times the Transform of the function itself. Notice, the Transform acts as an operator, when applied, replaces the derivative with a linear expression.

The rules of Integration

Similar to derivation rule the integration rule can be derived as follows,

$$L[\int_0^t f(t)] = \int_{0-}^{\infty} [\int_0^t f(t)] e^{-st} dt$$

$$L[\int_0^t f(t)] = [\int_0^t f(t) dt] \frac{e^{-st}}{-s} \Big|_{0-}^{\infty} - \frac{1}{-s} \int_{0-}^{\infty} f(t) e^{-st} dt$$

$$L[\int_0^t f(t)] = \frac{1}{s} L[f(t)] \tag{5.12}$$

The Laplace Transform of the integral of a function is $1/s$ times the Laplace Transform of the function itself as shown in Equation 5.12.

The table 5.1 describes Laplace Transform of some common functions. Notice some entries in the table have a multiplication factor of $u(t)$. This effectively provides a range of function values for $t < 0$, since the unit step function by definition has a value of 0 for $t<0$ and a value of 1 for $t >= 0$.

We will use the differential and integral rules extensively in our analysis of systems that produce response to a stimulus in the form a differential equation.

Inverse Laplace Transform

If you think of the Transform, as creating a spectrum of the frequencies in a system, then think of the inverse as bringing all the frequencies back into a single function. It is like reverting to the time based expression. The inverse operation is a complex integral, where all the frequencies in a system are summed together to recover the original time based function; the following integral describes the inverse operation,

$$x(t) = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} X(s) e^{st} \, ds \tag{5.13}$$

In practice however, a table of Transform pairs (such as the Table 5.1) derived in Equation 5.2 through 5.9 are being used to identify the corresponding time based function for a given Transformed expression.

Table 5.1. Laplace Transform of some common functions.

	$f(t)$	$F(s) = \int_{0-}^{\infty} f(t) e^{-st} \, dt$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$e^{-\alpha t} u(t)$	$\frac{1}{s+\alpha}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$\sin(t) u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$
6	$\cos(t) u(t)$	$\frac{s}{s^2 + \omega_0^2}$
7	$e^{\alpha t} \sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2 + \omega^2}$
8	$e^{\alpha t} \cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2 + \omega^2}$
9	$ae^{-\alpha t} \cos \beta t + \frac{(b - a\alpha)}{\beta} e^{-\alpha t} \sin \beta t$	$\frac{as+b}{(s+\alpha)^2 + \beta^2}$

Solving Differential Equations with Laplace Transform

The differentiation and integration rules of the Laplace Transforms, derived in Equation 5.10, 5.11 and 5.12, transform an ordinary differential equation into an algebraic expression. The expression can be simplified and solved for the dependent and independent variables, but the solution is still an expression in the form of a Laplace Transform. If a true solution is desired then the Transformed output must be converted to the time based function, either by using inverse Laplace integral or using a table lookup such as the Table 5.1 to identify the appropriate Transform pair.

You will soon realize that all the necessary design information is present in the Transformed output such as the system's Gain and Phase change at different frequency input etc. But, for now, we would like to show that the Transform indeed provides a solution of differential equation. One advantage of the Laplace Transform method is that the Transform is simply a frequency dependent linear function, ideally suited for analyzing the system's response at different frequencies, in-fact the original answer is not as important as the Transform plot of the magnitude vs. the frequency that can describe the system's behavior over the entire range of frequencies of interest. You will see the use of frequency response in the next section of Transfer Functions and the Bode plot.

Let's take the example of the impulse response of a series *RLC* circuit as described in the Figure 5.2. We have already discussed this example in previous chapter and solved the problem using the conventional method, and now we would like to use the method of substitution by Laplace Transform.

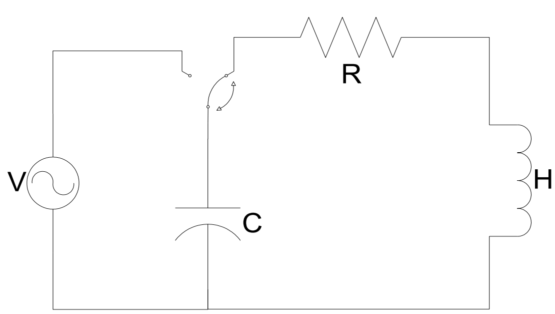
The relationship between the current and voltages across different elements in the circuits are,

$$i_C = C\frac{dv_C}{dt} \qquad i_R = \frac{v_R}{R} \qquad i_L = \frac{1}{L}\int v_L dt$$

$$v_C = \frac{1}{C}\int idt \qquad v_R = Ri_C \qquad v_L = L\frac{di_L}{dt} \qquad \frac{dv_L}{dt} = L\frac{d^2 i_L}{dt^2}$$

The Kirchoff's current and voltage law describes the relationship for the series network as,

$$i_C = i_R = i_L$$
$$v_C + v_R + v_L = 0$$



****** Insert Figure 5.2 here ******
Figure 5.2 Series *RLC* circuit

The second order differential equation is formed because of summing the voltages. Let *h* be the impulse response, resulting the following equation,

$$L\frac{dh}{dt} + Rh + \frac{1}{C}\int hdt = \delta(t) \tag{5.14}$$

Conventionally, the Transform is expressed in bold letters; so let *H(s)* be the Transform of the impulse response. The integrals and differentials of the Equation 5.14 are then reduced to the following algebraic form

$$L[L\frac{dh}{dt}] = LsH(s)$$

$$L[Rh] = RH(s)$$

$$L[\frac{1}{C}\int hdt] = \frac{1}{Cs}H(s)$$

The Laplace Transform of the impulse input from the Equation 5.2 we get,

$$L[\delta(t)] = 1$$

Substituting the Transforms into the terms of Equation 5.14, we get the following algebraic equations,

$$(Ls + R + \frac{1}{Cs})H(s) = 1$$

$$H(s) = \frac{R}{L}\frac{s}{s^2 + (R/L)s + 1/LC}$$

Substituting the coefficients with the standard notations,

$$\omega_0^2 = \frac{1}{LC} \qquad \alpha = \frac{R}{2L} \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2} \qquad \phi = \sin^{-1}\frac{\alpha}{\omega_0}$$

Solving for *H(s)*,

$$H(s) = \frac{R}{L}\frac{s}{(s^2 + 2\alpha s + \alpha^2) + (\omega_0^2 - \alpha^2)}$$

$$H(s) = \frac{R}{L}\frac{s}{(s+\alpha)^2 + (\omega_d^2)} \qquad (5.15)$$

Although we could perform the Inverse Laplace Transform of the Equation 5.15 to obtain the time domain function, but with some manipulation the result can be made to fit the pattern being described in the entry 9 of the Table 5.1.

$$h(t) = \frac{R}{L}e^{-\alpha t}(\cos\omega_d t - \frac{\alpha}{\omega_d}\sin\omega_d t) \qquad (5.16)$$

The solution of Equation 5.14 obtained through the Lapalce Transform matches the result of Equation 4.24 that was obtained through conventional method.

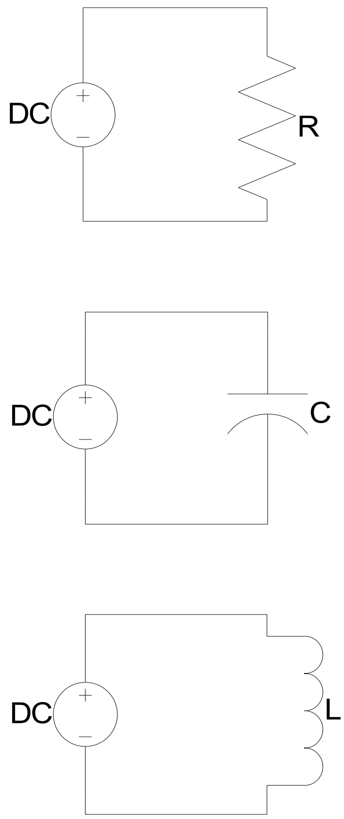
It is obvious that Laplace Transform could be used as an effective method of solving differential equations, but Engineering design is far from simply about solving specific problems, it is about modeling a system and analyzing the system's response to various input conditions. A Transfer Function of a system is a mathematical model that describes the system's response to input frequencies and establishes a relationship between the input and output (also known as the 'Gain') of the system. Even if a system is complex, (having multiple stages of input and output) still we can perform the transform on its subsystems and linearly add or multiply each subsystem at later stages to obtain the overall system response. That leads to the concept of impedance and admittance as the Transform of the lowest level of subsystems, meaning the component level of inductors, capacitors and resistors.

Impedance and Admittance

The lowest levels of subsystems in electrical networks are the individual resistors, capacitors and inductors and for a linear system the overall system response is the same as the sum of individual response. The Impedance and Admittance are the transforms of basic elements of such subsystems. In electrical networks the input and output are usually voltages or current. The Impedance is defined as the ratio of the Laplace Transform of the output voltage and the input current, while Admittance is defined as the ratio of the output current and input voltages.

When components are placed in series, their impedances are added and for elements connected in parallel, the admittance is added. We have already seen that for resistance in series the individual resistances are added and for resistance in parallel the admittance are added. The impedance and admittance are simply input and output relationship of voltage and current which is equally applicable to all elements including the resistors, capacitors and inductors.

The Figure 5.3.a, b and c describe the impedance and admittance of the three basic elements of an electrical network, namely, the resistors, capacitors and inductors.



$$v = Ri$$

$$v = \frac{1}{C}\int i dt$$

$$v = L\frac{di}{dt}$$

****** Insert Figure 5.3 here ******
Figure 5.3 Impedance and Admittance of a) resistors, b) capacitors, c) inductors

The impedance Z of a resistor is the Laplace Transform of the output voltage and the input current, while the admittance G is the Laplace Transform of the output current and the input voltage.

$$Z_R = \frac{L[v]}{L[i]} = \frac{V}{I} = R \qquad G_R = \frac{L[i]}{L[v]} = \frac{1}{R}$$

Similarly, the input current and output voltage and output current and input voltage across capacitors and inductors are defined in terms of the Laplace Transform as,

$$Z_L = \frac{L[v]}{L[i]} = \frac{V}{I} = sL \qquad G_L = \frac{L[i]}{L[v]} = \frac{1}{sL}$$

$$Z_C = \frac{L[v]}{L[i]} = \frac{V}{I} = \frac{1}{sC} \qquad G_C = \frac{L[i]}{L[v]} = sC$$

Applying the concept of impedance to the *RC* circuit of Figure 5.4.a, we could obtain the response with the following derivative,

The current through the loop is

$$I = V_{in} / (Z_R + Z_C)$$

The voltage across the resistor *R* is

$$V_{out} = \frac{V_{in}}{Z_R + Z_C} Z_R = \frac{V_{in}}{1 + \dfrac{1}{sRC}}$$

The overall system response is therefore,

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + {}^{1}\!/_{sRC}}$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{sRC}{sRC + 1}$$

As mentioned earlier the quantity $s = \sigma + j\omega$ is a combination of exponentially decaying factor and a sinusoidal function of frequency ω. Ignoring the exponent decay factor as merely initial transient response contributor, we can formulate the steady state response as a function of frequency ω as,

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{j\omega RC}{j\omega RC + 1}$$

or

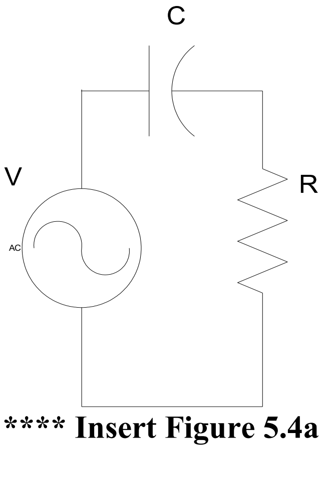
$$H(s) = \frac{V_{out}}{V_{in}} = \frac{j\omega RC}{1 + j\omega RC} \times \frac{1 - j\omega RC}{1 - j\omega RC} = \frac{j\omega RC + (\omega RC)^2}{1 + (\omega RC)^2}$$

Multiplying it with its complex conjugate, and taking square root gives the magnitude $|H(s)|$.

$$|H(s)| = \sqrt{H(s) \times {}^{*}H(s)} = \sqrt{\frac{(\omega RC)^2 + j\omega RC}{1 + (\omega RC)^2} \times \frac{(\omega RC)^2 - j\omega RC}{1 + (\omega RC)^2}}$$

$$|H(s)| = \sqrt{H(s) \times {}^{*}H(s)} = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \tag{5.17}$$

A plot of *H(s)* as the Gain vs. *s* (frequency variable) is presented in Figure 5.4.b for the values of *C=1* and *R=2* and a glance on the plot shows that at high frequencies (*s >> RC*) the 'Gain' is much higher compare to low frequencies (*s << RC*), essentially suppressing the low frequencies but letting the high frequencies pass through. In fact, at 0 frequency (meaning DC level) the output is essentially 0. Clearly, the circuit of Figure 5.4 is a high pass filter. Such analysis of system's response is possible only through the Transfer Function of a system.



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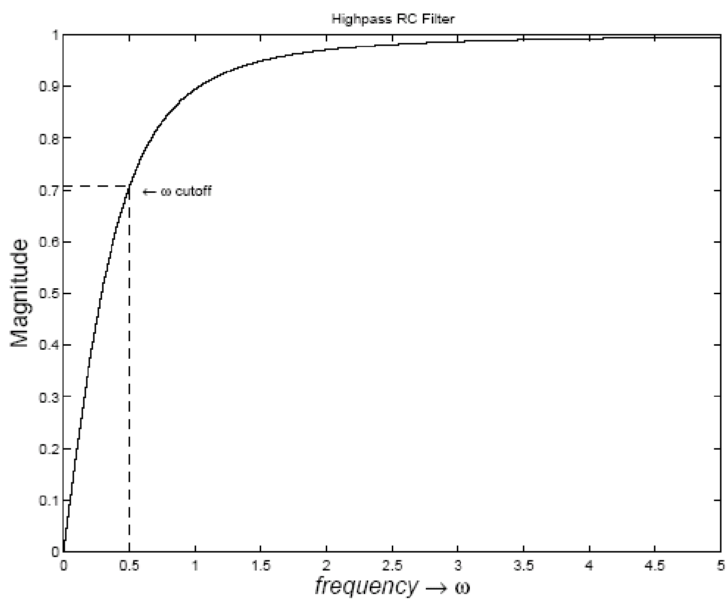

****** Insert Figure 5.4b here ******

Figure 5.4. a. RC circuit. b) Frequency response curve using linear scale.

A better way of presenting the frequency plot is through logarithmic scale, to highlight a large range of frequency values. The Figure 5.5 is the plot of the Equation 5.17, using the logarithmic scale for the X-axis.

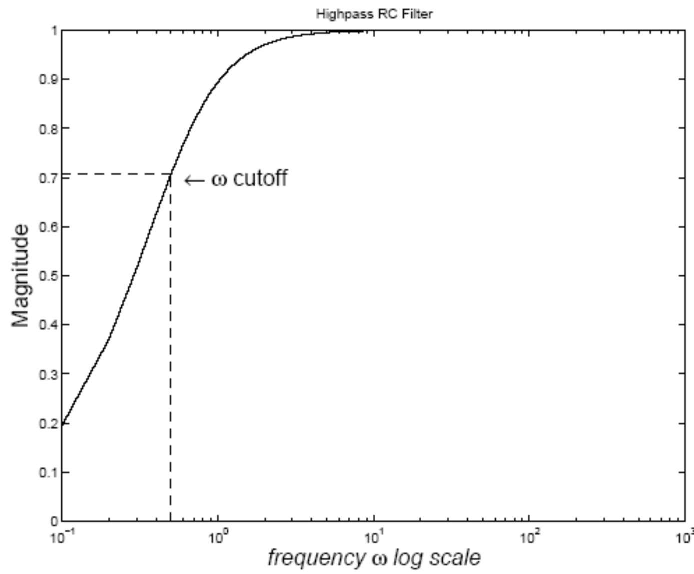

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Figure 5.5 Frequency response curve using logarithmic scale

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Transfer Function

The example of the high-pass filter presented in the previous section shows that the simplification of a differential equation using the Laplace Transform offers a great tool for analyzing a systems behavior. The algebraic relationship of input and output in the form of a systems response is ideal for describing a model for the system, especially, the frequency response of the system. We have already seen in Fourier Transform that any input to a system can be described in terms of its constituent frequencies, and having a model of frequency response in the form of Laplace Transform just about sums up everything in the nature that we need to analyze.

Although, the impulse response developed in the previous section can be used for the frequency analysis, that we can simply plot the magnitude as a function of frequency, but we seek a uniform representation that is applicable to all systems. For now, we focus our attention on how a Transfer Function helps us in analyzing the frequency response of a system and later in the section of poles and zeros and Bode plot we develop a uniform approach that is applicable to all types.

A Transfer Function of a system is the input and output relationship in the form of frequency response that is suitable for frequency analysis. In the context of Laplace

Transform a system with input voltage $V_{in}(s)$ and output voltage $V_{out}(s)$ *V(s)* having an impulse response *H(s)*, has the following Transfer function,

$$\frac{V_{out}(s)}{V_{in}(s)} = H(s)$$

The significance of the impulse response is that in the form of Laplace Transform it provides a frequency dependent algebraic relationship, just what we needed to analyze the systems behavior at various frequencies. Notice the difference between the Fourier Transform and Laplace Transform. The complex variable $s = \sigma + j\omega$ in the Laplace Transform includes the exponential decaying factor as well as the natural frequency of the system, whereas the Fourier Transform has only the natural frequencies of the system.

It is true that after some time, all systems follow the frequency of the input signals and the original transient exponent decay dies down eventually, making the σ as an insignificant component of *s*. It essentially states that the steady state Laplace Transform is the same as the Fourier Transform. Also, in real life, if we are dealing with only frequency function as input to a system, the exponent decay is insignificant and can be ignored for all practical purpose. The Transform is then reduced to a frequency response function only, where $s = j\omega$*,* as shown in the following equation,

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = H(j\omega)$$

Another conclusion we can draw is that the Convolution process (where impulse response is convolved with the input function) is now converted into multiplication operation, the Transform of the impulse response multiplied with the Transform of the input function produces the Transform of the output.

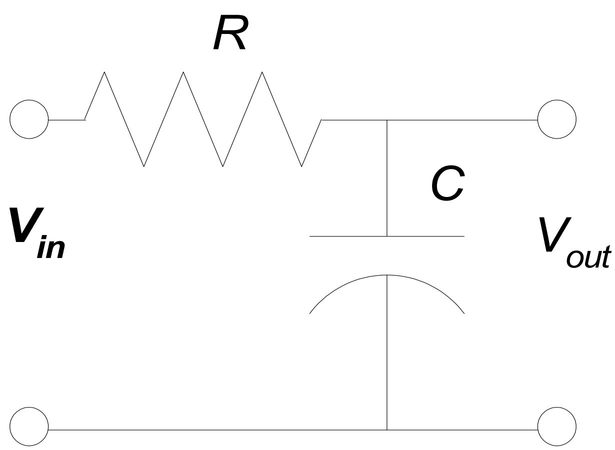
$$V_{out}(j\omega) = H(j\omega) \times V_{in}(j\omega)$$
$$v_{out}(t) = h(t) * v_{in}(t)$$

See chapter 4 for the description of Convolution process.

Filters, wether analog or digital play significant role in communication system design. Radio, Telephony, Voice synthesis all have filters to either extract and amplify or reject and subside a specific frequency or range of frequencies. A system with energy storage elements acts as a filter and the solution of the governing differential equation is the filter's response. The Laplace Transform of the response (commonly known as frequency response) is a model for the filter and provides a method for analyzing the behavior of the system. We have devoted a complete chapter on filter design, but for now, we will present a simple low pass filter as an example to show how the Transfer Function helps us analyze a system's response.

Filter Design and Transfer function

Consider the response of the simple *RC* circuit as shown in Figure 5.6. The governing differential equation was already solved, using both the conventional method as well as the Laplace Transform method. The following is the Transfer Function or the Impulse Response of the system where we have eliminated the exponential decaying factor from the variable $s = \sigma + j\omega$.



****** Insert Figure 5.6 here ******
Figure 5.6. A simple low-pass filter with one resistor and one capacitor

$$\frac{V_{out}}{V_{in}} = H(j\omega) = \frac{1}{1 + j\omega RC} \tag{5.18}$$

The Equation 5.18 is a complex number, whose real and imaginary components may be extracted by multiplying the numerator and denominator by $1 - j\omega RC$

$$\frac{V_{out}}{V_{in}} = H(j\omega) = \frac{1}{1 + j\omega RC} \times \frac{1 - j\omega RC}{1 - j\omega RC}$$

Thus the real component is

$$\mathrm{Re}\{H(j\omega)\} = \frac{1}{1 + (\omega RC)^2}$$

and the imaginary component is

$$\mathrm{Im}\{H(j\omega)\} = \frac{-j\omega RC}{1 + (\omega RC)^2}$$

The magnitude and phase as a function of frequency may be defined as,

The magnitude,

$$| H(j\omega) |= \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{\frac{1+(\omega RC)^2}{(1+(\omega RC)^2)^2}} = \frac{1}{\sqrt{1+(\omega RC)^2}} \tag{5.19}$$

And the phase,

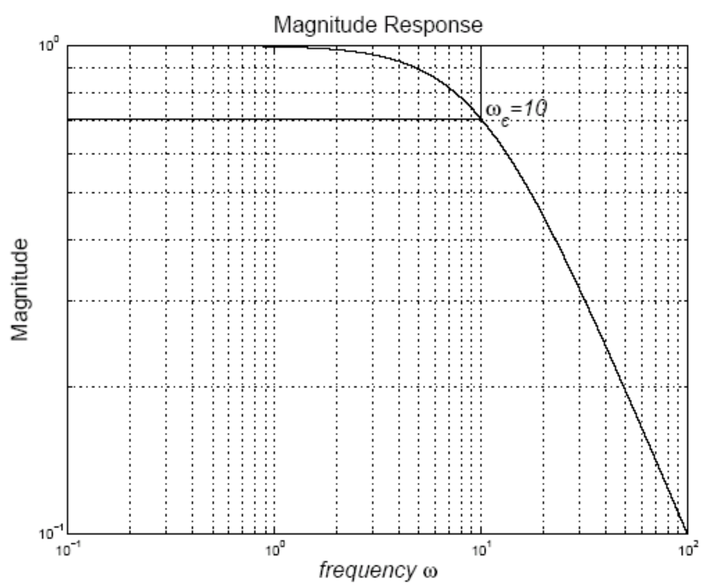
$$\angle H(j\omega) = \tan^{-1}(\frac{\mathrm{Im}}{\mathrm{Re}}) = -\tan^{-1}\omega RC = \theta$$

For circuit components of $C = 1\mu F$, $R = 1k\Omega$ the Equation 5.19 is reduced to,

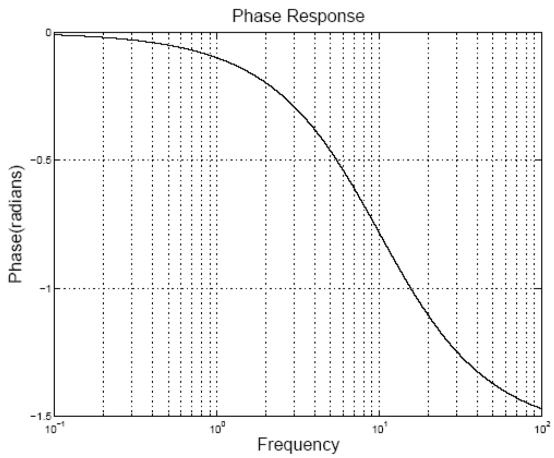
$$| H(j\omega) |= \frac{1}{\sqrt{1+\omega 10^{-3}}}, \theta = \tan^{-1}(\omega 10^{-3}) \tag{5.20}$$

$$\angle\theta = \tan^{-1}(\omega 10^{-3})$$

The Figure 5.7 is a plot of the magnitude and phase response of the Equation 5.20.



****** Insert Figure 5.7a here ******



****** Insert Figure 5.7b here ******

Figure 5.7.a. The magnitude response, b) The phase response of the low pass circuit of Figure 5.6.

For a given input frequency $Vin = A\sin(\omega_0 t + \phi)$, the output is computed as,

$$Vout = |H(j\omega)|\,Vin = \frac{A}{\sqrt{1+(\omega_0 RC)^2}}\sin(\omega_0 + \phi - \tan^{-1}\omega_0 RC)$$

Let's analyze the result of Equation 5.20 (using different input frequencies) for the sake of comparison, and see if there could be any uniformity in the way, the results are presented,

Input frequency $\omega = 1 rad/\sec$

$$|H(j1)| = \frac{1}{\sqrt{1+(1\times 10^{-3})}} \approx 0.99 \tag{5.21}$$

Input frequency $\omega = 10 rad/\sec$

$$|H(j10)| = \frac{1}{\sqrt{1+(10\times 10^{-3})}} \approx 0.99 \tag{5.22}$$

Input frequency $\omega = 100 rad/\sec$

$$|H(j100)| = \frac{1}{\sqrt{1+(100\times 10^{-3})}} \approx 0.95 \tag{5.23}$$

Input frequency $\omega = 1000 rad/\sec$

$$|H(j1000)| = \frac{1}{\sqrt{1+(1000\times 10^{-3})}} \approx 0.707 \tag{5.24}$$

Input frequency $\omega = 10000 rad/\sec$

$$|H(j10000)| = \frac{1}{\sqrt{1+(10000\times10^{-3})}} \approx 0.305 \tag{5.25}$$

As you can see there is hardly any affect on the magnitude when the input frequency is below $\omega = 1000 rad/\sec$ (see Equation 5.21 through 5.25). But then as the frequency is increased the gain is reduced considerably and at $\omega = 10000 rad/\sec$ the magnitude is reduced to only .305 of the original input (see Equation 5.25). The circuit can easily be qualified as a low-pass filter that would not disturb input frequencies below $\omega = 1000 rad/\sec$, but then it would start subsiding the higher frequencies above $\omega = 1000 rad/\sec$. There is a clear trend in the output. Up until the Equation 5.24, the magnitude remains unaffected, (only dropping to 0.707) but then there is a steep drop beyond that point. You will see later in the section of poles and zeros that the mark of 0.707 is chosen as a turning point in the frequency response plot and is being used as thumb rule in design practice.

So far, we have studied systems with very few energy storage elements. In an *RC* circuit there was only one (the capacitor) and in an *RLC* circuit there were barely two elements, the inductor and the capacitor. It was not too difficult to analyze them by the conventional means of algebra, but it was obvious that a more complex system would produce a more complex response. The following is a method to reduce a complex solution to a manageable algebraic form.

Reducing a Transfer Function's complexity

In general, a Transfer Function of a more complex system can be described as a ratio of two polynomials,

$$H(s) = \frac{P(s)}{Q(s)} = \frac{z_0 s^m + z_1 s^{m-1} + ..z_{m-1}s + z_m}{p_0 s^n + p_1 s^{n-1} + ..p_{n-1}s + p_n}$$

Where *P(s)* is numerator polynomial in variable *s* and coefficient *z* and *Q(s)* is a denominator polynomial in variable *s* with coefficient *p*.

An alternate way of writing the polynomials in factored form is,

$$H(s) = \frac{P(s)}{Q(s)} = K\frac{(s-z_1)\times(s-z_2)\times..\times(s-z_m)}{(s-p_1)\times(s-p_2)\times..\times(s-p_n)} \tag{5.26}$$

The numerator coefficients z_i are called zeros of the Transfer Function and p_i are called the poles of the Transfer Function, whereas the *K* is a real number scaling factor (Equation 5.26). The terminology **poles** and **zeros** are used for an apparent reason that at $s=z_i$ the numerator becomes zero, making the Transfer Function a zero and at $s=p_i$ the denominator becomes zero, making the Transfer Function an infinite value, making the graph look like a pole of a tent –if there was a graph of *H(s)*. But this is just a fallacy, you can not subtract p from s or z from s, s is an imaginary number and p or z may be real or imaginary, they have to be handled with the rules of complex number algebra. The poles and zero have other significance and that will become apparent in the next section.

Poles, Zeros and Steady state Frequency Response

When a Transfer Function is presented in the form of a numerator and denominator polynomial (as shown in Equation 5.26), each term in parenthesis is a complex number vector, the z_1 and p_1 could be real or complex number and the quantity s is equal to $\sigma + j\omega$. For a steady state frequency response, the quantity s may be replace with $j\omega$ by ignoring the exponential decay σ and the Equation 5.26 can be written as,

$$H(j\omega) = \frac{P(s)}{Q(s)} = K\frac{(j\omega - z_1)\times(j\omega - z_2)\times..\times(j\omega - z_m)}{(j\omega - p_1)\times(j\omega - p_2)\times..\times(j\omega - p_n)} \tag{5.27}$$

The plot of the Transfer Function of the Equation 5.27 is useless as it is discontinuous at each pole location; instead, a plot of Transfer Function magnitude vs. the frequency is being used to describe the relationship between the response and the input frequency. A more appropriate way of describing the Equation 5.27 is,

$$|H(s)| = K\frac{(\sqrt{\omega^2 + z_1^2})\times(\sqrt{\omega^2 + z_2^2})\times..\times(\sqrt{\omega^2 + z_i^2})}{(\sqrt{\omega^2 + p_1^2})\times(\sqrt{\omega^2 + p_2^2})\times..\times(\sqrt{\omega^2 + p_i^2})}$$

The significance of **pole** and **zeros** is that at specific frequency of $\omega = p_i$ (called pole frequency) the magnitude of the Transfer Function drops down to 0.707 of the magnitude at the start frequency of $\omega = 0$, similarly at frequency $\omega = z_i$ (called zero frequency) the Transfer Function magnitude is raised by 1.404 of the magnitude at the start frequency. The poles and zeros are only a convenient way of identifying the turning points in the magnitude of the Transfer Function, since, at frequencies less then the poles and zero frequencies, there is hardly any impact on the magnitude, but at frequencies above the poles and zeros there is a rapid drop in the magnitude (in case of poles) and rapid rise in magnitude in case of zeros. (The poles and zeros are essentially vectors, whereas pole and zero frequencies are only indicative of the turning point in magnitude of the response.)

Let's take the simple case where z_i and p_i are real. The Transfer Function of the low-pass *RC* filter of the previous example was already in the form of Equation 5.21 and had no zero, a pole at $j\omega = 10^3$ and the Gain factor $K=10^3$.

$$H(j\omega) = 10^3 \times \frac{1}{(j\omega + 1000)} \tag{5.28}$$

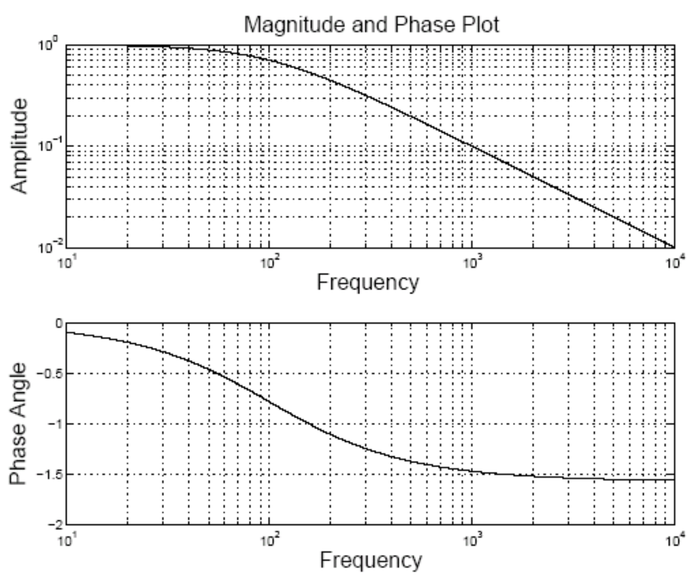
Calculating the magnitude of the Transfer Function at different frequencies,

$\omega = 0, \omega = 100, \omega = 1000 and \omega = 10000$

$$H(j0) = 10^3 \times 1 \Big/ \sqrt{0^2 + 1000^2} = 1$$
$$H(j100) = 10^3 \times 1 \Big/ \sqrt{100^2 + 1000^2} = 0.995$$
$$H(j1000) = 10^3 \times 1 \Big/ \sqrt{1000^2 + 1000^2} = 0.707$$
$$H(j10000) = 10^3 \times 1 \Big/ \sqrt{10000^2 + 1000^2} = 0.1$$

Notice, at the pole frequency ($\omega = 1000$), the 'Gain' is 0.707 of its peak magnitude.

A plot of the frequency response is shown in Figure 5.8.a and the Figure 5.8.b is the diagram showing pole vector of the Equation 5.28.



****** Insert Figure 5.8 here ******
Figure 5.8. A plot of the frequency response

It should be noted that when poles and zeros are complex numbers their net effect on the magnitude is little distorted. The complex vectors always come in conjugate pairs. The tails lie not at the real axis but somewhere in the region $-\alpha + j\omega$ and $-\alpha - j\omega$ as shown in Figure 5.9.b. Let's consider the parallel RLC circuit of Figure 5.9.a who's Transfer Function is given as,

$$H(s) = \frac{1}{C}\frac{s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \tag{5.29}$$

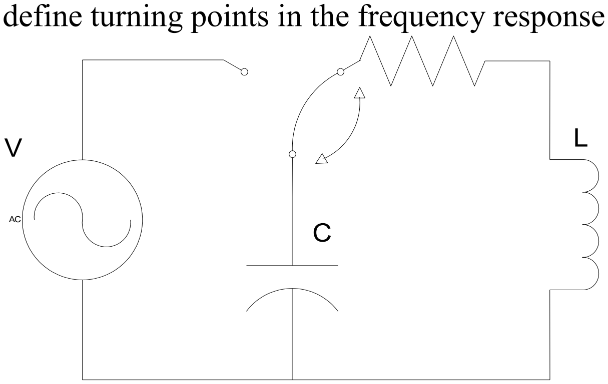
Substituting $\alpha = \frac{1}{2RC}, \omega_d = \sqrt{\omega_0^2 - \alpha^2}, \omega_0^2 = \frac{1}{LC}$ we can rewrite the Equation 5.29 in the factored form as,

$$H(s) = \frac{1}{C}\frac{s}{[s-(-\alpha + j\omega_d)][s-(-\alpha - j\omega_d)]} \tag{5.30}$$

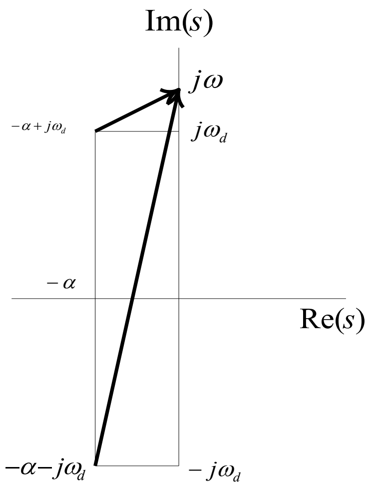
The Equation 5.30 has one zero at s=0 and two poles $s = (-\alpha + j\omega_d)$ and $s = (-\alpha - j\omega_d)$. With the more appropriate form of magnitude response, the Equation 5.30 is simplified as,

$$|H(j\omega)| = \frac{1}{C}\frac{j\omega - 0}{[j\omega-(-\alpha + j\omega_d)][j\omega-(-\alpha - j\omega_d)]}$$

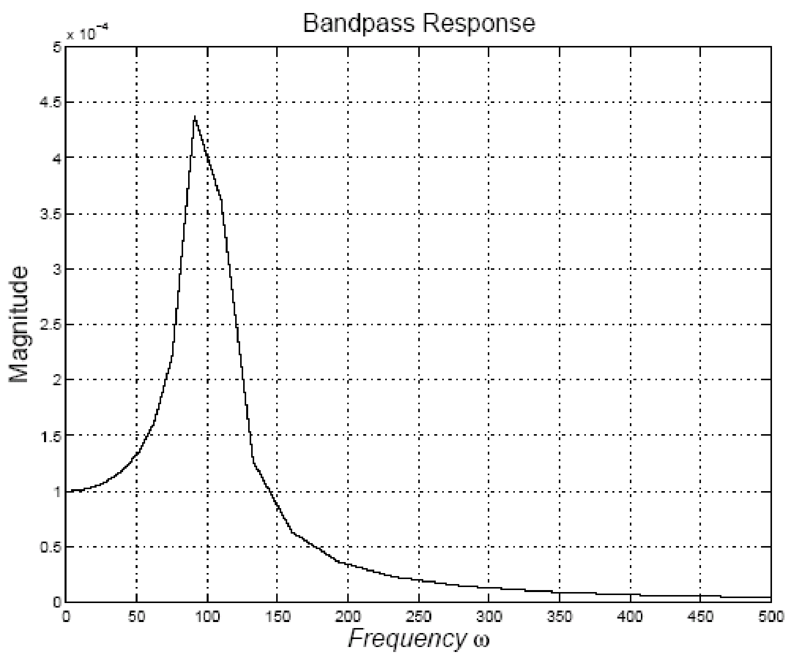
The frequency response is shown in Figure 5.9.c and the vector magnitudes ($s-(-\alpha + j\omega_d)$ and $s-(-\alpha - j\omega_d)$) are plotted in the Figure 5.9.b. The combined effect of the conjugate pair is such that the overall gain is more at the pole frequencies then at 0 frequency, (enhancing a band of frequencies and suppressing the rest). It is obvious that we can select circuit components (*RLC*) such that at certain frequencies, the gain is much higher compare to the rest of the frequencies, a property that is exploited in the design of band-pass filters in the next chapter. Thus, a rule of thumb is devised that poles and zeros define turning points in the frequency response of a system.



****** Insert Figure 5.9a here ******



****** Insert Figure 5.9b here ******



****** Insert Figure 5.9c here ******

Figure 5.9.a. Series *RLC* circuit Figure 5.9.b. The pole diagram of a complex conjugate pole vectors, Figure 5.9.c. A plot of the frequency response of the

We will discuss poles and zeros further in the next section but for now let's talk about the range of numbers we are dealing with. The frequencies of interest can vary from as low as 0 to as high as mega hertz. A linear scale certainly will not do the job. It is easier to describe the magnitude and phase response of the system using logarithmic scale, as it broadens the view of the response plot. Taking logarithm of both sides of the Equation 5.27 we get,

$$| H(j\omega) |=| K | \times | j\omega - z_0 | \times | j\omega - z_1 | \times ... \times \frac{1}{| j\omega - p_0 |} \times \frac{1}{| j\omega - p_1 |}$$

$$\log | H(j\omega) |= \log | K | + \log | j\omega - z_0 | + ... + \log \frac{1}{| j\omega - p_0 |} + ... + \log \frac{1}{| j\omega - p_i |} \quad (5.31)$$

The logarithmic scale not only enhances the range of numbers that we can deal with but also converts the multiplication operation into an addition operation, simplifying the operation of computing the Transfer Function. Next, we will discus **decibel** as the logarithmic scale that has been accepted as a standard procedure to quantify Transfer Functions.

Decibel dB

The Transfer function being a ratio of output and input, such as voltage or current, is a Dimensionless ratio. The magnitude is either less than unity, indicating a loss, or greater than unity indicating gain. The decibel (abbreviated dB) is a special unit being described as, 10 times the log of the ratio of output power, delivered by the system and the power given to the system, as shown in Equation 5.23.

$$\text{Power gain} = 10\log\frac{P_{out}}{P_{in}} = 10\log\left[\frac{(v_{out}^2/R_{out})}{(v_{in}^2/R_{in})}\right] = 20\log\left|\frac{v_{out}}{v_{in}}\right| + 10\log\left|\frac{R_{out}}{R_{in}}\right| \quad (5.32)$$

Since, we always try to match the input and output impedance of the system, the last quantity in Equation 5.32 is considered 0 and the gain is usually described as,

$$20\log\left|\frac{v_{out}}{v_{in}}\right|.$$

When we speak of Transfer Function in terms of 'dB', we talk about, 20 times the log of the magnitude of the Transfer Function. Thus, a Gain of 10, is 20dB, Gain of 100 is 40 dB, Gain of 1000 is 60 dB, Gain of 0.707 = -3dB etc. With the help of decibel (as our unit of magnitude), we can describe quantities as small a number as 0.0001, all the way to 1000000, with only 10 division on a plot. You will see some examples shortly, but one last thing in terms of displaying the plot of the magnitude of the Transfer Function vs. frequency and that is the Bode plot.

Bode plot

A bird's eye view of the system's frequency response can easily be calculated with a fair degree of approximation, if we consider what happens to the gain at the turning point of its poles and zeros. Considering the example of $H(j\omega)=10^3\times\dfrac{1}{j\omega-(-1000)}$ and the dB value of the Gain and the phase at various frequency, shown in the Table 5.2

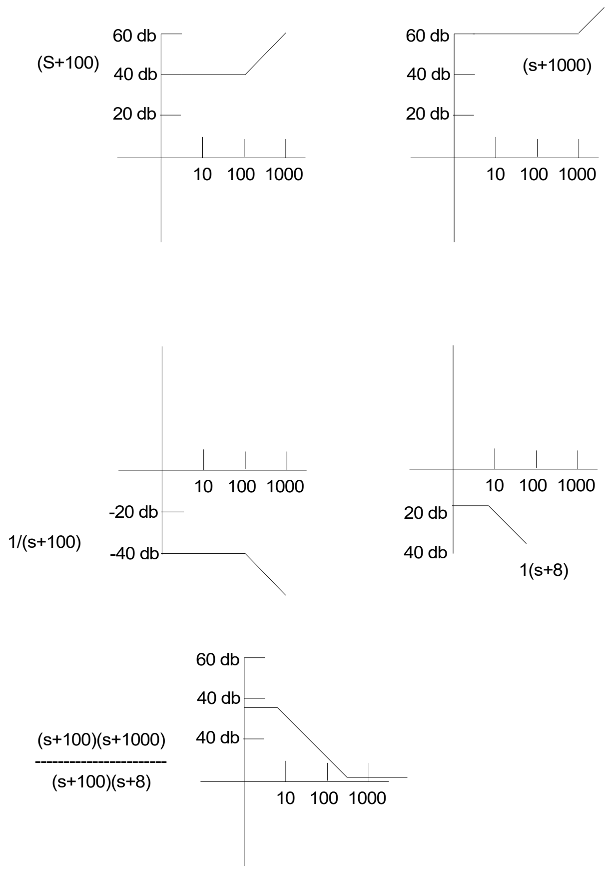
Table 5.2. Magnitude and phase of the Transfer Function of the expression $\dfrac{1}{j\omega-(-1000)}$

ω	$\lvert H(j\omega)\rvert$	$\angle H(j\omega)$	20 log$\lvert 1000\rvert$	$20\log\left\lvert\dfrac{1}{j\omega+1000}\right\rvert$	$20\log\lvert H(j\omega)\rvert$
1 rad/sec	0.99	0 deg	60 dB	-60 dB	0 dB
10 rad/sec	0.99	0 deg	60 dB	-60 dB	0 dB
100 rad/sec	0.95	5.7 deg	60 dB	-60 dB	0 dB
1000 rad/sec	0.707	45 deg	60 dB	-63 dB	-3 dB
10000 rad/sec	0.1	85 deg	60 dB	-80 dB	-20 dB
100000 rad/sec	0.01	89 deg	60 dB	-100 dB	-40 dB
1000000 rad/sec	0.001	90 deg	60 dB	-1000 dB	-60 dB

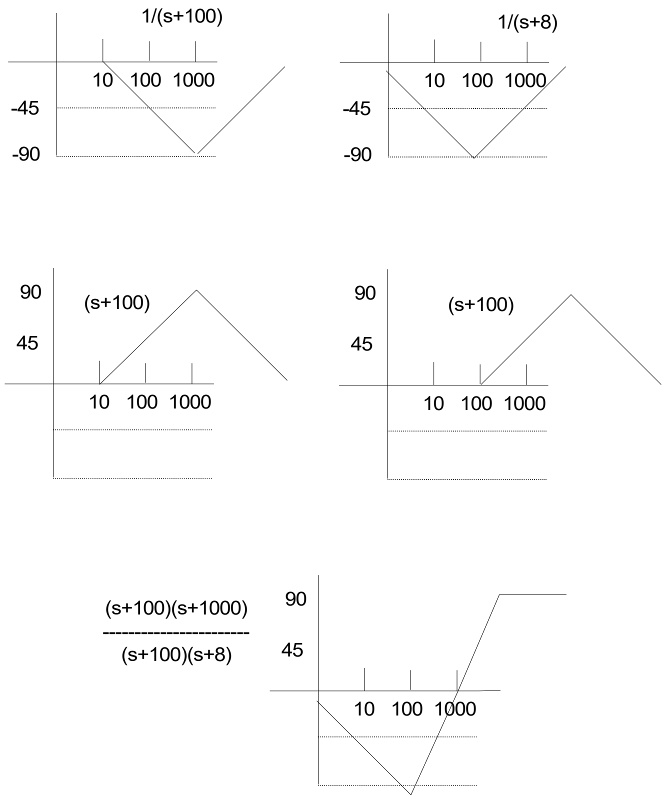
The contribution of poles towards the Gain of the Transfer Function (Column 5 of the Table 5.2) is such that at frequencies less then the pole frequencies, the slope is approximately equals to 0 (basically a flat response), but frequencies above the turning point (the pole frequency), the slope is approximately -20dB per decade. Similarly, for zeros, the slope is flat for frequencies less then the ‘zero’ frequency, but anything beyond that the slope is +20dB per decade.

With the facts mentioned above, we can take individual poles and zero and sketch out the independent response and linearly adding them all later to obtain the final response. The steps are highlighted in the Figure 5.10 for the following example Transfer Function of two poles and two zeros,

$$H(s)=\frac{(s+100)(s+1000)}{(s+100)(s+8)}$$



****** Insert Figure 5.10a here ******



****** Insert Figure 5.10bhere ******

Figure 5.10. Bode plot of poles and zeros

Once the frequency response is identified that meets the requirements, the design job is over. The poles and zeros are the design criteria as they specify the values of the circuit components you need to fulfill the design. There are standard circuits that fit each poles and zeros and you will see the examples later when we design analog filters. With this, we conclude the discussion of Laplace Transform. Next, we discuss the digital counterpart of the Laplace Transform the z-Transform.

The z-Transform

The z-Transform is the digital domain counterpart of the analog domain Laplace Transform. So far, all the mathematics of the solutions of differential equations revolved around the behavior of the physical components of resistors, capacitors and inductors. But in real life, the problems are defined in terms of response and the challenge is to design the circuit that match the response. Suppose, the problem is to remove all the frequencies above 20 KHz from an incoming signal, then, the solution would be to pick the low pass filter circuit and simply chose the component values that match the Transfer Function of the circuit. Computers could do all that just as easily, only if they were fast enough. If it weren't for the speed problem, we would probably never use the physical components in our systems. After all, every memory location of the computer could be used as an energy storage device, making it possible to create system that is thousand times more complex.

There is a correlation between the speed of execution and the input frequency of the digital world. The speed affects the sampling rate and the sampling rate determines the maximum input frequency the digital system can process. Just to give you a perspective; A 20 KHz sound wave (the best human can hear) requires a sampling period of 25 microseconds. That means, within this time interval, the computer must take the A/D converter reading, multiply with the necessary coefficients (as if it was solving a differential equation) and produce the output to a D/A converter. Modern day computers can handle this task easily, so the audio frequency is pretty much within the range of the digital domain. But, for a radio frequency of mega hertz range, this kind of processing is out of the question (at least for now). If you suspect there is a high frequency noise in your system, which is above the sampling rate, you must add a hardware filter to eliminate the noise before taking any data.

You will see shortly that the z-Transform of a difference equation, gives us an expression in frequency domain, similar to the Laplace Transform. The expression can be used to analyze the system's behavior in response to various inputs. Since, engineering design is not about solving the actual difference equation but to find the system's response to various inputs, the Transfer Function is the mathematical way to express such a response.

In the digital world, the differential equation of the analog domain becomes the Difference equation of the discrete time. The time t of the continuous time corresponds to discrete time $k\Delta t$ (where k is the sample number and Δt is the data acquisition time per sample). The frequency in radians per sec ω_t is now normalized with $\omega_s = \omega_t \Delta t$ (radians per second times seconds per sample) and instead of integral over the time interval dt, we use summation over the interval radians per sample and instead of integrating from 0 to infinity the range is from 0 to the number of samples in two times the Nyquist frequency. With these new terminologies, we replace the Laplace Transform with the following definition of the z-Transform.

$$\int_{0-}^{\infty} f(t)e^{-st}\,dt \rightarrow \sum_{k=0}^{\infty} f(k)e^{-j\omega_s k}$$

$$Z[f(k)] = F(z) = \sum_{k=0}^{\infty} f(k)z^{-k} \tag{5.33}$$

Where $z = e^{j\omega}$ and *f(k)* is the *kth* sample value.

Similar to Laplace Transform the z-Transform converts a time dependent function into a frequency dependent function, but the frequency instead of radians per second is now in radians per sample $\omega_s = \omega_t \Delta t$. Think of the z-Transform as a spectrum of all the frequencies present in the sample *f(k),* up to the Nyquist frequency. In general, the z-Transform of the input *x(k)* is defined as $X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$ and the output *y(k)* as $Y(z) = \sum_{k=0}^{\infty} y(k)z^{-k}$.

Inverse z-Transform

Technically, if the z-Transform is a spectrum of all the frequencies in a sample, then the inverse z-Transform should give you the sample with all the component frequencies in it, and the following integral accomplishes the reconstitution of the original sample.

$$x(n) = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz \tag{5.34}$$

It is not necessary to recover the original function using the inverse z-Transform; we could simply rearrange the Transform in the form of recognizable poles and zeros and use the table lookup method similar to Laplace Transform. The table 5.3 is a table of z-Transform of some commonly used functions.

Our main goal in solving difference equation through the z-Transform is to develop a relationship between the input and output, in the form of a Transfer Function. Essentially, defining an expression for the impulse response of the system.

The followings are some useful z-Transforms.

Unit Impulse function

The impulse function is defined only at *k=0,*

$$Z[\delta(k)] = \sum_{k=0}^{\infty} \delta(k) z^{-k} = \delta(0) = 1 \tag{5.35}$$

Unit Step function

A closed form expression can be derived using the sum of a geometric sequence

$$\sum_{k=0}^{\infty} c^k z^{-k} = \frac{1}{1 - cz^{-1}} \tag{5.36}$$

Substituting for c=1 in Equation 5.36, for a unit step function we get,

$$Z[u(k)] = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}}$$

Exponent functions

Substituting for $c=a^k e^{jkb}$ in Equation 5.36 we get,

$$Z[a^k e^{jbk}] = \sum_{k=0}^{\infty} a^k e^{jbk} z^{-k} = \frac{1}{1 - ae^{jb} z^{-1}} \tag{5.37}$$

Sin function

Using exponential form of the sin function we get,

$$Z[\frac{1}{2} a^k (e^{jbk} - e^{-jbk})] = \sum_{k=0}^{\infty} a^k e^{jbk} z^{-k} = \frac{1}{1 - \frac{1}{2} a(e^{jbk} - e^{-jbk}) z^{-1}}$$

$$Z[\frac{1}{2} a^k (\sin b)] = \frac{(a \sin b) z^{-1}}{1 - 2az^{-1} + a^2 z^{-2}} \tag{5.38}$$

Cos function

Using exponential form of the cos function we get,

$$Z[\frac{1}{2} a^k (e^{jbk\Delta t} + e^{-jbk\Delta t})] = \sum_{k=0}^{\infty} a^k e^{jbk\Delta t} z^{-k} = \frac{1}{1 - \frac{1}{2} a(e^{jbk\Delta t} + e^{-jbk\Delta t}) z^{-1}}$$

$$Z[\frac{1}{2} a^k (\sin b)] = \frac{(1 - az^{-1} \cos b\Delta t)}{1 - 2az^{-1} \cos b\Delta t + a^2 z^{-2}} \tag{5.39}$$

Table 5.3. Commonly used z-Transforms

	$f(k)$	*X(z)*

1	$\delta(k)$ *(impulse function)*	1
2	$u(k)$ *(unit step)*	$\frac{1}{1-z^{-1}}$
3	c^k *(exponential)*	$\frac{1}{1-cz^{-1}}$
4	*k (ramp)*	$\frac{z^{-1}}{(1-z^{-1})^2}$
5	$a^k\cos(bk\Delta t)$ cosine function	$\frac{(1-az^{-1}\cos(b\Delta t))}{(1-2az^{-1}\cos(b\Delta t)+a^2z^{-2})}$
6	$a^k\sin(bk\Delta t)$ sin function	$\frac{(a\sin b)z^{-1}}{1-2az^{-1}+a^2z^{-2}}$

The Difference rule

The most important property of the z-Transform is its Difference rule. Similar to the Laplace Transform that changed a differential into a multiplication operation (see the derivative of Equation 5.10) where the first derivative is *s* times the Transform and the second derivative is s^2 times the Transform and so on, with the same context the z-Transform also changes the first difference into z^{-1} and second difference into z^{-2} and so on, as shown in the following derivative,

$$Z[y_k - y_{k-1}] = \sum_{k=0}^{\infty}(y_k z^{-k}) - \sum_{k=0}^{\infty}(y_k z^{-(k-1)})$$

The summation in the second term can be expressed as,

$$Z[y_{k-1}] = z^{-1}\sum_{k=0}^{\infty} y_k z^{-k}$$

$$Z[y_{k-2}] = z^{-2}\sum_{k=0}^{\infty} y_k z^{-k}$$

The Difference rule is also known as the shift property of the z-Transform, as multiplying with z^{-1} shifts the transform to the previous input and multiplying with z^{-2} shifts to the previous to previous input etc.

Next, we will see how the difference rule helps us in defining an algebraic relationship of the output and input in the form of a Transfer Function.

The Transfer Function in z-Transform

A general difference equation can be written in the following way,

$$y_0(k) + b_1 y(k-1) + ...b_m y(k-M) = a_0 x(k) + a_1 x(k-1) + .. + a_n x(k-N)$$

Applying the difference rule to the above equation produces a polynomial in the power of *z*,

$$z^0 Y(z) + b_1 z^{-1} Y(z) + ...b_m z^{-m} Y(z) = z^0 a_0 X(z) + a_1 z^{-1} X(z) + .. + a_n z^{-N} X(z) \qquad (5.40)$$

The Equation 5.40 can be simplified as,

$$Y(z)\left(1 + \sum_{m=1}^{M} b_m z^{-m}\right) = X(z) \sum_{n=0}^{N} a_n z^{-n} \qquad (5.41)$$

From Equation 5.41 we get a Transfer Function *H(z)* showing a relationship of the output *Y(z)* and the input *X(z)* as follows,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{n=0}^{N} a_n z^{-n}}{1 + \sum_{m=0}^{M} b_m z^{-m}} \qquad (5.42)$$

The Expression in Equation 5.42 is a rational polynomial in variable z. The expression is suitable for plotting the magnitude *H(z)* as a function of sampling frequency (radians per sample). This is similar to the Transfer Function of continuous time as a rational polynomial in *s*. The designers of the system could use the plot to see, for any given input frequency what the output magnitude would look like. A simple example is the process of averaging input values to eliminate the ripples in the input data. This is like implementing a low pass filter. In this scheme, every new input *x(k)* is added to the previous input *x(k-1)* and divided by 2 to produce the output *y(k)*, An iterative computer algorithm could be developed to produce the output as shown below,

$$y_k = \frac{1}{2}(x_k + x_{k-1})$$

Taking the Transform of both sides,

$$Y(z) = (\frac{1}{2} X(z) + \frac{1}{2} z^{-1} X(z))$$

To determine the frequency response, we obtain the Transfer Function as,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} + \frac{1}{2} z^{-1}$$

Setting $z = e^{j\omega\Delta t}$,

$$H(z) = \frac{1}{2} + \frac{1}{2} e^{-j\omega\Delta t} \rightarrow \frac{1}{2} + \frac{1}{2}(\cos(\omega\Delta t) - j\sin(\omega\Delta t))$$

The magnitude of the Transfer Function is obtained from multiplying with its complex conjugate,

$$| H(z) | = \sqrt{\{0.5(1+\cos(\omega\Delta t))\}^2 - \{0.5 j(\sin(\omega\Delta t)\}^2} \qquad (5.43)$$

Simplifying the term in Equation 5.43, using the identity $\sin^2(\omega t) + \cos^2(\omega t) = 1$,

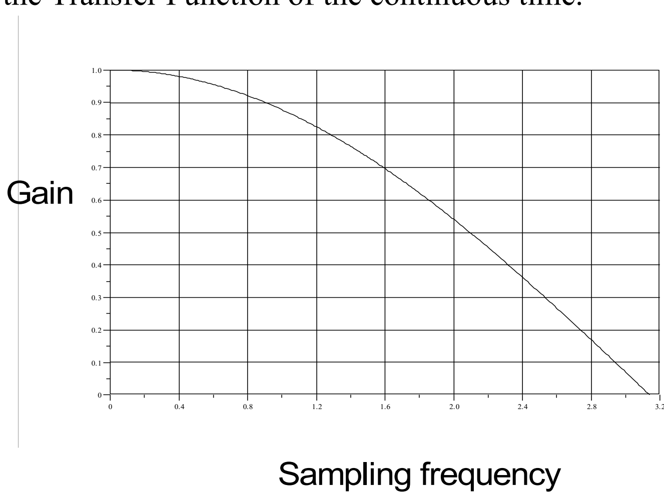
The Transfer Function magnitude is,

$$| H(z) | = \sqrt{0.5 + 0.5\cos(\omega\Delta t)} \qquad (5.44)$$

And the phase change is,

$$\Phi_H(\omega\Delta t) = \arctan(\frac{\sin(\omega\Delta t)}{(1+\cos(\omega\Delta t)} \qquad (5.45)$$

The magnitude of the Equation 5.44 is plotted in Figure 5.11, and you can see that the algorithm does suppress high frequencies very well. The example is given only to express the usefulness of the Transfer Function plot, albeit a very simple one. If the system is complex and involves several terms in the polynomials, such as the one in Equation 5.40, a better way to represent the Transfer Function is in the form of poles and zeros, just like the Transfer Function of the continuous time.



****** Insert Figure 5.11 here ******

Figure 5.11. The averaging filter, Gain vs. Sampling frequency.

Poles and Zeros of z-Transform

A rational polynomial can be expressed as either a product or a sum of a partial fraction expansion, with real and imaginary coefficients. Presenting the polynomial as a product generates poles and zeros, similar to the continuous time Transfer Function. The terms in the poles corresponds to the frequency in radians per sample where the system gain is dropped to -3db of its unit magnitude and the terms in zeros corresponds to rise in gain of 3db. Since π radians per sample are the maximum frequency in the z-Transform, the terms in poles and zeros are indicative of fraction of maximum sampling frequency and must lie within a unit circle. Plotting poles and zeros locations on a unit circle provide useful information regarding the frequency response of the system.

We will make extensive use of the poles and zeros and the Transfer Functions in the next chapter of Filter Design, but next is an example to prove the point that z-Transform is essentially a solution of difference equation.

Solving difference equation with z-Transform
Let's work through the example of the *RC* network as shown in Figure 5.12 for the component values of *R*=1, *C*=1 and $\Delta t = 0.1$ sec per samples, and develop its discrete time impulse response using Convolution and then z-Transform,

Using x and y as the input and output, the difference equation is

$$RC\frac{dy}{dt} + y = x$$

$$RC\frac{y_k - y_{k-1}}{\Delta t} + y_k = x_k$$

$$y_k = \frac{y_{k-1}}{1 + \Delta t/RC} + \frac{\Delta t/RC}{1 + \Delta t/RC}x_k$$

Using the approximation $(1 + \Delta t/RC)^{-1} \approx 1 - \Delta t/RC$ and neglecting the higher order terms, we get,

$$y_k = a_0 x_k + b_1 y(k-1) \tag{5.46}$$

Where $a_0 = \Delta t/RC, b_1 = (1 - a_0)$

The impulse response of the Equation can be obtained by injecting a sample of unit impulse,

$$x(k) = \delta(k)\begin{cases} 1....k = 0 \\ 0....k \neq 0 \end{cases}$$

$$y(0) = h(0) = a_0$$
$$y(1) = h(1) = a_0 b_1$$
$$y(2) = h(2) = a_0 b_1^2$$
$$\vdots$$
$$y(k) = h(k) = a_0 b_1^k$$
$$y(k) = a_0 x(k) + a_0 b_1 x(k-1) + a_0 b_1^2 x(k-2) + \ldots + a_0 b_1^k x(0) \tag{5.47}$$

The same result can be obtained through expanding the z-Transform of the Equation 5.46 as shown in the following transformation,

$$y_k = a_0 x_k + b_1 y(k-1)$$
$$Y(z) = a_0 X(z) + b_1 z^{-1} Y(z)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0}{1 - b_1 z^{-1}} \tag{5.48}$$

We can use the analogy of a geometric sequence, whose expansion is given as,

$$\frac{1}{1 - cz^{-1}} = 1 + c^1 z^{-1} + +c^2 z^{-2} + \cdots + c^n z^{-n}$$

The result of the Equation 5.32 can be expanded as,

$$Y(z) = a_0 X(z) + a_0 b_1^1 z^{-1} X(z) + a_0 b_1^2 z^{-2} X(z) \cdots a_0 b_1^k z^{-k} X(z) \tag{5.49}$$

Taking the inverse of the Equation 5.49,

$$y(k) = a_0 x(k) + a_0 b_1 x(k-1) + a_0 b_1^2 x(k-2) + \ldots + a_0 b_1^k x(0) \tag{5.50}$$

The result of the Equation 5.50 matches the Equation 5.47. Both of them are solutions of difference equation derived in Equation 5.46.

Our main purpose of going through the rigor of mathematics to find the solutions of differential equations is that they form the basis for realizing our main objective, meaning, designing the analog and digital filters and that is being discussed in the next chapter.

Conclusion

The goal in this chapter was to derive a closed form expression of a solution of differential equation called *Transfer Function*, describing the relationship between the input frequencies and output response. The poles and zeros extracted from the Transfer Function completely describe a system's response to varying input frequencies, thus helping in establishing a relationship between the inputs and outputs of a linear system.