Chapter4　Solutions of Differential Equations

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Introduction

So far we have studied Fourier analysis as a way to observe an event and analyze its constituent frequencies. Now we consider the processing of a signal, meaning, integrating, differentiating, smoothing and filtering of input signals to produce a desired outcome. You will soon realize that many of the processing in a digital signal processing meant implementing solutions of differential equations, whether it is a temperature control system or speech analysis or vibration studies. The cause and effect has relationship that lends itself to the formulation of differential equations. We encounter them when we design analog electrical circuits and mechanical systems and since the digital signal processing has its roots in the analog signal processing, it is imperative to understand the methodology of deriving the algorithms to solve differential equations.

In this chapter we will devote our attention to the Convolution process, as the method of solving a differential equation, while the other technique of Laplace Transfer will be differed till the next chapter. If the goal is to design a simple integrator or a differentiator then we only need to derive a difference equation to be implemented as an iterative algorithm in a digital computer, but implementing a digital or analog filter requires deriving a closed form expression called *Transfer Function.* A transfer function defines the output of a system, as a function of frequency, indicating what frequencies will be suppressed while others are available without degradation. We begin our study with a refresher and establish the necessary mathematical foundation.

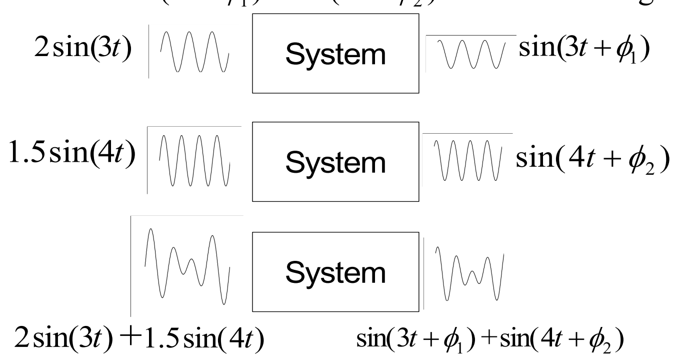
Linear Time Invariant System

A system is considered linear if its output is directly proportional to its input, such as current and voltage relationship in an electrical system (an example of a non linear system is a relationship between current and power). The time invariance condition describes a system in which a delay in the input causes same amount of delay in the output. The solutions that are presented in this chapter require the system to be **Linear and Time Invariant**.

A Linear System has the **Additive** property and a **Homogeneity** property. An additive system is where the response to a sum of inputs is the same as the sum of the individual responses and a system is homogenous when the scaling of the input by some amount also results in the scaling of the output by the same amount (a sinusoidal input remains a sinusoidal output without affecting the frequency of the input signal, only the magnitude and phase may change). It should be noted that we will be dealing with only Linear Systems and the differential equations would be linear differential equations.

A system will exhibit a certain response, depending upon the input energy applied to the system. If the response is due to the stored energy such as a charge on a capacitor or current in the inductor, then it is the **natural response** of the system. But if the response is due to some external energy source then it is the **forced response** of the system.

The condition of a system being Linear is not very strict. All we are asking is that, if the forced response of the system is being studied there must not be any prior force present in the system and if the natural response of the system is under study the input force must have been removed after giving the initial push. The other *additive* and homogeneity property requires that the quantity under study must have a simple one to one relationship between the input and output of the system. A system exhibits additive property if the output of several individual inputs is the same as output of each independent input summed together. For example, if the response of the input $2\sin(3t)$ is $\sin(3t+\phi_1)$ and $1.5\sin(4t)$ is $\sin(4t+\phi_2)$ then in a linear system the response of $2\sin(3t)+1.5\sin(4t)$ would be $\sin(3t+\phi_1)+\sin(4t+\phi_2)$ as shown in Figure 4.1.



****** Insert Figure 4.1 here ******

Figure 4.1 Showing additive property of a linear system, the combined output is the same as individual output summed together.

In other words if the input is decomposed into several distinct excitations and the output is a superimposed result of each individual excitation, the system is additive in nature. The **Time Invariant** condition refers to the fact that if the input excitation is delay by *t* amount of time then the output will also be delayed by the same amount of time.

Differential Equations

A differential equation is formed when the function and the rate of change of the function (or the derivative) appears together in the same algebraic equation. The *order* of a differential equation is defined as the highest order of the derivative contained in it. A **first order differential equation** is formed in an electrical network if there is only one energy storage element in the circuit, such as a capacitor or an inductor. The current across the capacitor is proportional to the rate of change of the voltage applied, $i = C\dfrac{dv}{dt}$.

Similarly, the voltage across an inductor is proportional to the rate of change of the current applied, $v = L\frac{di}{dt}$. (Resistors are not considered energy storage elements.) A **second order differential equation** is formed when there are two different energy storage elements in a circuit, such as a capacitor and inductor in series or parallel, resulting in a second derivative in the system. The current across a parallel combination of a capacitor and an inductor is $i = C\frac{d^2v}{dt^2} + \frac{1}{L}v$, while voltage across a series combination of a capacitor and an inductor is $v = L\frac{d^2i}{dt^2} + \frac{1}{C}i$.

Systems Response

A system, which is capable of storing energy such as an electrical circuit or mechanical system responses to a stimulus in two different ways, **natural response** and a **forced response**. The behavior determined by the internal energy storage elements is the natural response and the behavior determined by an external force is the forced response. Think about the energy stored when a spring is stretched. The spring may be forced to vibrate at any frequency by applying an external alternating force, but if the spring is stretched and released it will vibrate at its characteristic frequency determined by the specific spring constant. The natural response is the response due to the stored energy being released at natural pace. A system will always exhibit its natural response once the input excitation is removed from the system. Even if we don't remove it, as in case of obtaining a forced response, we can think about the input as a series of impulses applied and derive a solution as if several natural responses occurring sequentially. This is the basis for Convolution and will be discussed later in the chapter. It is easier from analysis point of view that we study the two responses independently and combine the result at our convenience at a later time. The system is guarantied not to alter the behavior since we are studying linear systems only.

Solutions of differential equations

Finding a response to an input excitation requires one to provide a **homogenous solution** as well as a **particular solution.** The homogenous solution is for the homogenous differential equation of the system formed as a result of applying the basic laws of physics on the circuit components of the system and it is devoid of an external force. It is also the natural response of the system as it is the result of solving equation when the input force is no longer in action. It is like seeing the residual effect due to the energy stored in the system. A particular solution of a differential equation is any function that satisfies the given differential equation. We will see that the particular solution is essentially the forced response of the system and the homogenous solution is the natural response of the system.

Linear Differential Equations

A linear differential equation is formed in a linear time invariant system as a result of meeting certain initial condition criteria. For the forced response of the system there must not be any force before the time $t < 0$ and for the natural response, the input force must be gone after time $t \geq 0$. These are the initial condition requirements for a differential equation to be linear. The initial condition for the natural response is also called **zero input condition** and for the forced response is the **zero state condition**. The forced response of a system that already has an input force at $t < 0$ creates a non-linear differential equation and it is not of interest to us from Digital Signal Processing point of view.

The homogenous solution and the particular solution can be derived separately for a linear system and the final result may be obtained by a simple addition of the two solutions.

We can solve differential equations using Convolution method or the Laplace Transform method. Both are equally important in their own respect and have usefulness in different applications. Convolution is the primarily tool in image processing while Laplace Transform is being used mainly in signal processing such as speech and controls systems.

The Laplace transform essentially converts a time domain signal into frequency domain and produces a response that is best suited for frequency analysis. Analog filters are implemented for accepting a desired frequency range as well as rejecting an unwanted frequency band from input signals. They are also being used during speech compression and transmission as they can be designed to produce a desired waveform. The Laplace transform is ideal for such Filter designs. Convolution on the other hand is a time domain process and is being mainly used in operations such as smoothing and filtering of input data.

The challenge in digital signal processing is to find a discrete time solution of a differential equation that has a counterpart continuous time solution. The theory of Laplace Transform is the basis for frequency domain analysis and through that the z Transform is being derived, which forms the basis for a discrete sample solution. Similarly, Convolution is the technique for solving time domain continuous time differential equations and there is a companion discrete time solution, suitable for software algorithms of Digital Signal Processing.

Filters, whether analog or digital are implemented as close form expressions of differential equation solutions.

The general form of a differential equation

An nth order differential equation is defined as

$$a_0 y + a_1 \frac{dy}{dt} + a_2 \frac{d^2 y}{dt^2} + \cdots + \frac{d^n y}{dt^n} = x(t)$$

Where $x(t)$ and $y(t)$ represent the input and output respectively and a_n are the constant coefficients describing the characteristics of the system elements. If the input excitation is removed i.e. $x(t) = 0$ then the resulting equation is called *homogenous* differential equation and requires only a general solution, otherwise, it is an *inhomogeneous* equation and requires a particular solution. It should be noted that a complete solution is a general solution plus a particular solution.

In the scope of digital signal processing, only the solutions of the first and the second order linear differential equation are considered. Other higher forms can be represented as cascaded or parallel combinations of simple first and second order equations. A solution of exponent form will always satisfy any order linear differential equation.

First order differential equations

The general form of differential equation with one energy storage element is given as,

$$a_0 y + a_1 \frac{dy}{dt} = f(t) \qquad (4.1)$$

Where y is the output and x is the input to the system.

Natural response

Removing the input source at time t =0 the input x becomes zero at $t \geq 0$ and the differential equation becomes,

$$a_0 y + a_1 \frac{dy}{dt} = 0$$

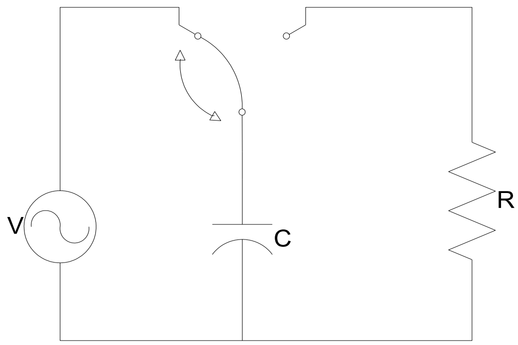
We will assume a solution of exponent form,

$$y = Ae^{kt}$$

$$\frac{dy}{dt} = kAe^{kt}$$

Selecting a value of $k = -\frac{a_1}{a_0}$ satisfies the Equation 4.1

The coefficient *A* can be solved using the initial condition of the system and for that let's work through an example. The capacitor C=1uF in the Figure 4.2 was initially charged with voltage V_0. At time *t*=0 the switch was then thrown in the position of the resistor R=1k, essentially removing the power source from the system. For the circuit component as given, we will determine the voltage as a function of time after the capacitor is connected with the resistor and also find the discrete time solution of the network for the sampling rate of 10 samples per second.



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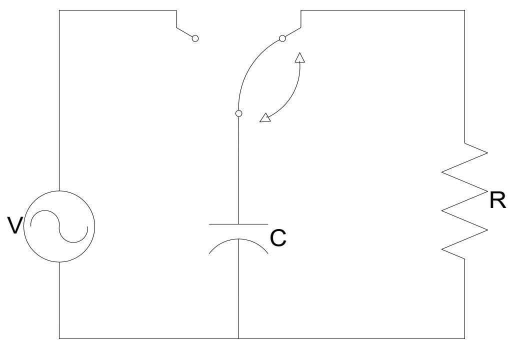


****** Inser Figure 4.2b here ******

Figure 4.2. Circuit showing an RC network. a) The capacitor is connected to the power source. b). The power is switched off at $t \geq 0$.

Assuming at t=0 the switch was thrown towards the resistor. Applying the Kirchoff's voltage law on the RC network we get a first order differential equation as a result of equating the voltages,

$$v_C(t) = v_R(t)$$

Using Kirchoff's current law of summing the current at a node, we get,

$$i_C(t) + i_R(t) = 0$$

The voltage and current across the resistor is,

$$v_R = Ri_R \qquad i_R = \frac{v_R}{R}$$

The voltage and current across the capacitor is,

$$v_c(t) = V_0 + \frac{1}{C}\int i dt \qquad i_c = C\frac{dv_C}{dt}$$

We get a homogenous equation by summing the two current

$$C\frac{dv_C}{dt} + \frac{v_C}{R} = 0$$

$$RC\frac{dv_C}{dt} + v_C = 0$$

Assuming v_C as an exponential function of time,

$$v_C = Ae^{kt} \qquad \frac{dv_C}{dt} = kAe^{kt}$$

Substituting the expressions into the homogenous solution

$$RCkAe^{kt} + Ae^{kt} = 0$$

$$(RCk+1)Ae^{kt} = 0$$

$$k = -\frac{1}{RC}$$

$$v_h = Ae^{-t/RC} \qquad (4.2)$$

The Equation 4.2 is the homogenous solution of a first order differential equation. To find the value of A we apply the initial condition of the voltage $v=V_0$ at $t=0$.

$$Ae^0 = V_0 \qquad A = V_0$$

Substituting the value of A and k in the expression we obtained the relationship of the voltage as a function of time as the solution of homogenous differential equation,

$$v_H(t) = V_0 e^{-t/RC} \qquad (4.3)$$

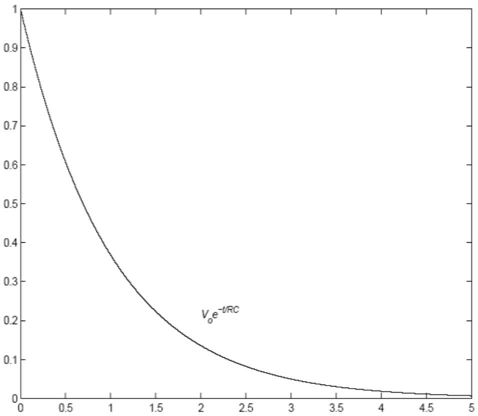
The particular solution is for the response due to the input force applied after the time t=0 and for our example we have 0 input force thus,

$$v_p = 0 \qquad (4.4)$$

The complete solution is obtained by adding homogenous solution (Equation 4.3) and the particular solution (Equation 4.4),

$$v_t = V_0 e^{-t/RC}$$

Figure 4.3 is the plot of the voltage as a function of time for an initial voltage of 1V, R = 1k and C=1uF.



****Insert Figure 4.3 here ****

Figure 4.3. The plot of the voltage as a function of time for the circuit of Figure 4.2b.

Forced Response

The natural response of a circuit will remain same regardless of the input applied, but the response to an external source depends entirely on the input type. Although the study of a response to an arbitrary input is our goal but we would derive the solution through the study of **step response** and the **impulse response**. An impulse is considered a special case of the step (with a very short duration) and an arbitrary input could be considered as an input of series of impulses very close to each other, albeit scaled by some magnitude and delayed by some time. If we know the response to a unit impulse, we can always find the response to an arbitrary input by considering the input as a series of impulses. Computing the cumulative affect of the previous responses and adding them to the output of the new response we can find the complete response. The process is known as **Convolution** and is simply a multiplication and addition operation; the multiplication is for scaling the input to the unit impulse and addition for taking into account the previous output. The frequency response is of special interest, as it would help us design circuits that act as frequency filters.

Step Response

Applying a constant current *I* at the instance t = 0 is synonym to a step input. The switch in the Figure 4.4 when thrown in the position of the capacitor will let the current flow into the *RC* circuit. We can imagine the response conceptually that the charge on the capacitor will slowly build up until it reaches the level of the voltage that appears on the Resistor *R* as well. Mathematically we can calculate the time it takes for the capacitor to reach the final voltage (v=*RI*) and that would be the final response of the system for a step input.

Applying the Kirchoff's Current Law on the circuit,

$$C\frac{dv}{dt}+\frac{1}{R}v=I$$

Where I indicate the constant current through the RC system.
The homogenous solution remains the same as described in the Equation 4.2

$$v_H = Ae^{-t/RC}$$

The particular solution is obtained by the new input condition of the constant current and we get the following particular solution

$$v_P = IR$$

You can verify the answer by placing the derivative $\frac{dv}{dt}=0$ into the general differential equation. The complete solution is obtained by adding the homogenous solution and the particular solution,

$$v = v_H + v_P$$

$$v = Ae^{-t/RC} + IR \tag{4.5}$$

The coefficient A is obtained from the initial condition

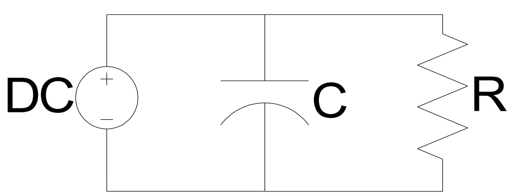
$$v(0)=0$$

$$Ae^0 + IR = 0 \qquad A=-IR$$

Substituting the value of A into Equation 4.5 we obtain the voltage as a function of time,

$$v = IR(1-e^{-t/RC}) \tag{4.6}$$

The Figure 4.4b is the plot of the Equation 4.6 showing the exponential rise of the voltage until it reaches the near full value of $v=RI$. The time constant T for an RC circuit is defined as the time it takes for the voltage to rise up to 0.63 of its full value and after the $4T$ the voltage reaches 99.98% of the full value.



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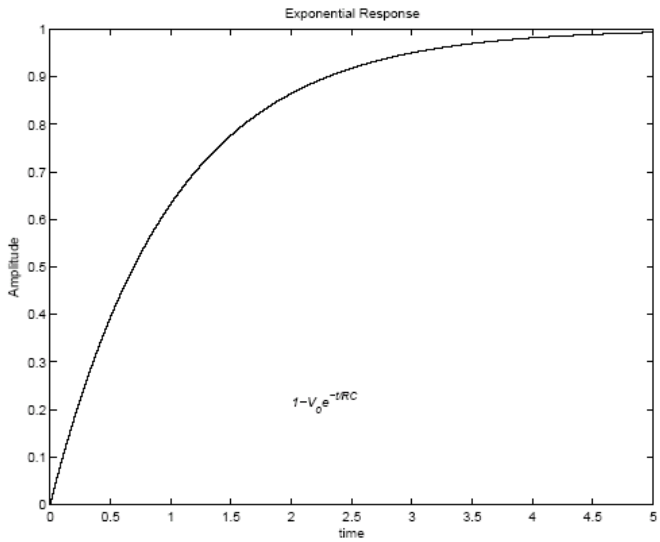

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Figure 4.4a.*RC* network for the Equation 4.6. b) The step response of the first order differential equation solution of the Equation 4.6 for *I=1* and *R=1*.

Discrete time solution

The discrete time sampling meant taking inputs regularly at a predetermined interval. The term is implied in Digital Signal Processing as it takes a finite amount of time for a processor to acquire and process the data before the next sample is being obtained. If the sampling rate of the input signal is $1/\Delta t$ samples per second, then the equivalent continuous time t for the *kth* sample is $k\Delta t$. To derive a discrete time solution of Equation 4.1 we use the method of backward differences,

$$RC\frac{y_k - y_{k-1}}{\Delta t} + y_k = x_k$$

Where x_k is the current input, y_k is the current output and $y_{(k-1)}$ is the previous output and the differential equation is simply a difference equation. Solving for y_k, we have,

$$y_k = \frac{1}{1+\Delta t/RC} y_{k-1} + \frac{\Delta t/RC}{1+\Delta t/RC} x_k$$

If the Δt is sufficiently small the term $(1+\Delta t/RC)^{-1}$ may be approximated as,

$$(1+\Delta t/RC)^{-1} \approx 1 - \Delta t/RC$$

Using a new term for $a_0 = \Delta t/RC$ and $b_1 = 1 - \Delta t/RC$, we get,

$$y_k = a_0 x_k + b_1 y_{k-1} \qquad (4.7)$$

Using the input voltage x=1V, RC=1 and sampling period $\Delta t = 0.1$, we get,

$y_k = 0.1x + 0.9y_{k-1}$

The general solution described in the Equation 4.7 is suitable for an iterative software algorithm for a digital computer implementation. Figure 4.5 shows the comparison between the discrete time solution and the continuous time solution of Equation 4.1 and Equation 4.7.

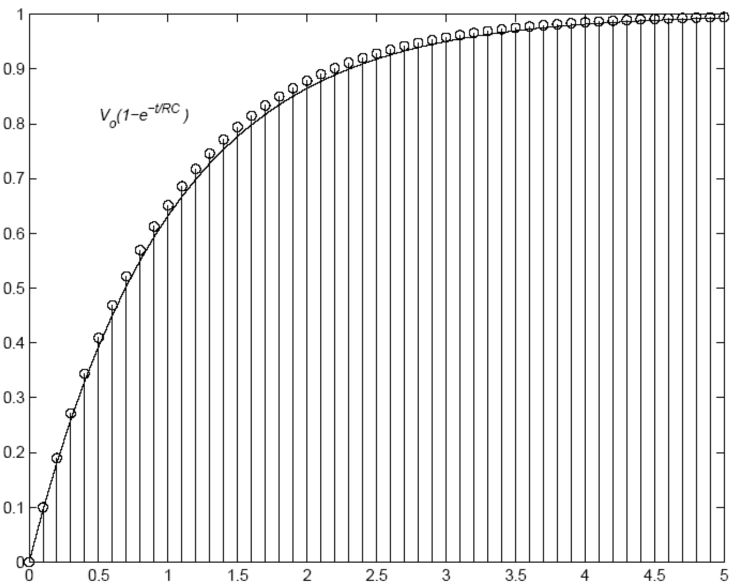

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Figure 4.5. Comparison of the discrete time solution and the continuous time solution for the circuit of Figure 4.2.

Unit Impulse Response

By definition, a unit impulse is a short duration pulse with a total area equal to one. The Figure 4.6 depicts such a function with a width of δ and a height of $\frac{1}{\delta}$ and is denoted by the symbol $\delta(t)$. It may not be possible to produce such a function in practice but conceptually it would helps us formulate the response of a continuous function, for we can think of a continuous function of time as a series of impulses.

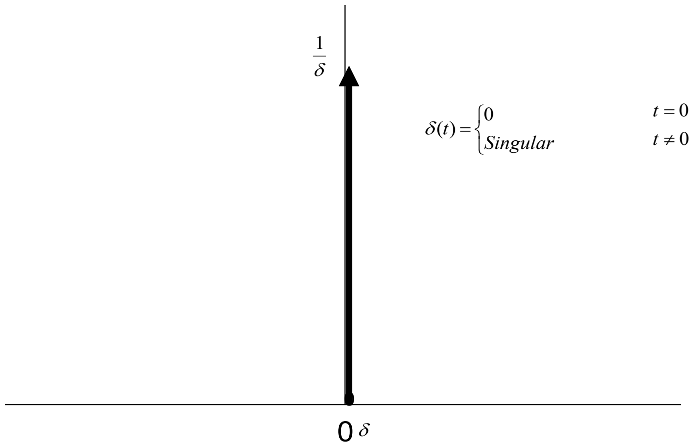

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Figure 4.6. An impulse function of width δ and a height of $\frac{1}{\delta}$

In order to find a solution of the homogenous equation for a unit impulse input, we need to find the amount of energy stored in the energy storage elements of the circuit, (such as capacitors and inductors) due to the impulse applied. Consider the charge build up in a capacitor due to the current function whose area equals one,

$$\int idt = CV = 1$$

$$V = \frac{1}{C}$$

Similarly, the current stored in an inductor due to voltage spike function whose area equals one,

$$\int vdt = LV = 1$$

$$V = \frac{1}{L}$$

Suffice to say that the initial condition produced by an impulse current on a capacitor is a voltage $V_0 = {}^1\!/_C$ and the current stored in an inductor is $I_0 = {}^1\!/_L$.

The requirement is to solve the solution of the following differential equation,

$$C\frac{dv}{dt} + \frac{1}{R}v = \delta$$

The particular solution is given as

$$v_P = \delta \qquad \text{at } t = 0$$

For all practical purpose the function $\delta = 0$ at time $t > 0$ making the particular solution

$$v_P = 0 \qquad \text{at } t > 0$$

Using the above-mentioned initial conditions we can obtain the complete response by adding the homogenous and the particular solution just like we did with the step input solution.

$$v = v_H + v_P$$

$$v = Ae^{-t/RC} + \delta \qquad \text{at } t = 0 \qquad (4.8)$$

The coefficient A is obtained from the initial condition

$$v(0) = V_0 = 1/C$$

$$v = Ae^0 + \delta = 1/C \qquad\qquad A = 1/C - \delta$$

Substituting the value of A into Equation 4.8 we obtain the voltage as a function of time,

$$v = (1/C - \delta)e^{-t/RC} + \delta \qquad \text{at } t = 0$$

Substituting $\delta = 0$ at time $t > 0$, we get the complete solution (or the response) to an impulse input in Equation 4.9.

$$v(t) = \frac{1}{C}e^{-t/RC} \tag{4.9}$$

Notice the similarities between the natural response (see Equation 4.2) and the impulse response of Equation 4.9. The comparison shows that the impulse response is essentially the same as the natural response of the system. The impulse function is being used only to visualize the amount of energy one can transfer in one go, without violating the initial condition of the zero input response. It may not be physically possible to provide such a force in reality, but it simplifies the mathematics. Once we determine the response of an impulse function, it is easier to derive the response of an arbitrary input function and that will be explained in the Convolution process later in the chapter. The Figure 4.7 is the plot of the Equation 4.9 as $\Delta t \to 0$.

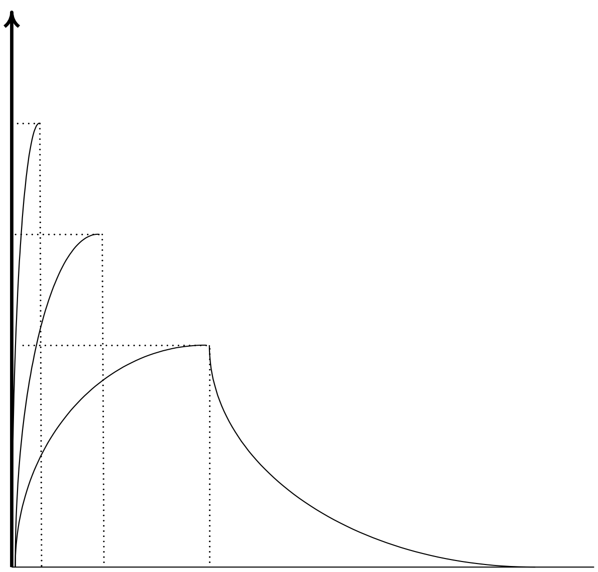

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Figure 4.7. The impulse response of the first order differential equation as $\Delta t \to 0$

Scaled Impulse Response

The impulse function doesn't have to be a unit area function. If the height is being halved the area will be halved and the output would be simply half of what a unit impulse response is.

$$\int idt = CV = 0.5$$

$$V = \frac{0.5}{C}$$

$$v(t) = \frac{0.5}{C} e^{-t/RC}$$

In other words if the input is scaled by a factor λ the output would be

$$v(t) = \frac{\lambda}{C} e^{-t/RC}$$

Arbitrary Input and Convolution

It is easier to obtain a solution for differential equations if the input is a well-defined mathematical function such as a step, an impulse or a sinusoidal input. Simply find the first and second derivative, plug-in the values and solve for the equations and get the output response. But an arbitrary input has no well-defined shape and form; a data acquisition system reading a channel value has no notion of the value being read. Events happen without set mathematical values. Take temperature and pressure for example. In order to design a control system for them, we must be able to predict the behavior of the system for an arbitrary input signal. In other words find the response of the system to an arbitrary input.

One way to analyze such an input is to look through the window of an impulse. If we breakdown the input as if it is a series of scaled impulses the job gets easier, as shown in Figure 4.8. We already know how to get the impulse response for an impulse function (see the Equation 4.9 for an *RC* network). Now it is just a matter of finding the scaling factors and getting the scaled responses. Then simply add the individual responses (of course delayed by some time) and we have the desired outcome. This is Convolution.

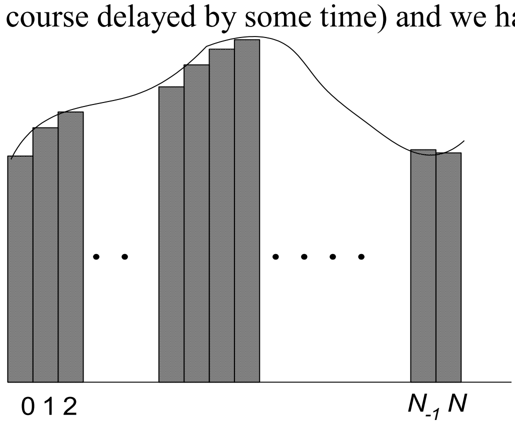

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Figure 4.8. Approximating an input as a series of impulses

The Convolution Process

The first step in Convolution is to isolate the impulse from the rest of the input and then scale it. This would create a trail of unit impulse scaled by the input signals at specific instance of time as shown in Figure 4.9. Without making it sound too complicated, if you think about it, the whole process is akin to simply taking the instantaneous values of the input signal at a specific interval of time. The value being acquired is the scaled unit impulse value, but of course delayed by the sampling interval. In this scheme, each new response will have contribution from the previous response that we must take into consideration into the current output.

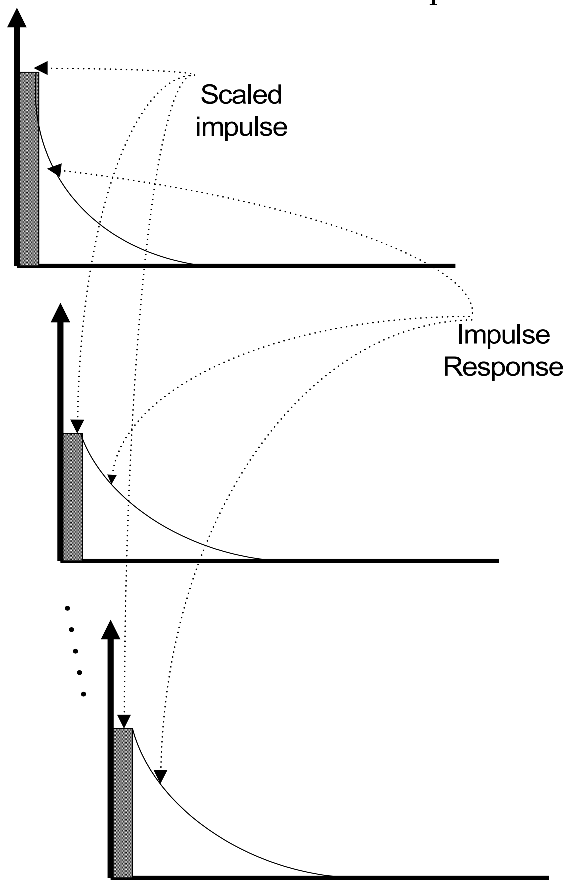

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Figure 4.9. Input sliced into a series of impulse and the response

Discrete Time Convolution

Mathematically, we can express the operation of discrete time sampling as shown in the Figure 4.9. A trail of areas delayed by the interval Δt. If t and $i_s(t)$ are the instantaneous sample time and sample value and t_n and $h(t_n)$ are the n_{th} sample time and unit impulse response then the n_{th} delayed response is $i_s(t_n)h(t-t_n)\Delta t$. The one before that is $i_s(t_{n-1})h(t-t_{n-1})\Delta t$, all the way to the beginning $i_s(t_0)h(t \;\; -t_0)\Delta t$. What we are doing is going back in time and finding the response of the previous sample again but this time using the next part of the impulse response. Every time you multiply the current input with the current impulse response you must add it to it the previous sample value multiplied by the delayed impulse response, see Figure 4.10 for a graphical description. Convolution process is the mathematical operation of accumulating the current response plus all the previous responses,

$$v(t_n) = i_s(t_0)h(t-t_0)\Delta t + i_s(t_1)h(t-t_1)\Delta t + ... + i_s(t_{n-1})h(t-t_{n-1})\Delta t$$

$$v(t_n) = \sum_{k=0}^{n-1} i_s(t_k)h(t-t_k)\Delta t \qquad (4.10)$$

Taking the limit as $\Delta t \to 0$ the summation in Equation 4.10 becomes an integral as shown in Equation 4.11.

$$v(t) = \int_{t_0}^{t} i_s(t_k)h(t-t_k)dt \qquad t \geq 0 \qquad (4.11)$$

We can obtain the discrete time equivalent of the Equation 4.10 by substituting n_{th} impulse response and $t_k\Delta = k_{th}$ input sample as shown in Equation 4.12. Notice the sign of the convolution operator *x(n)*h(n)*, a multiplication followed by the addition.

$$x(n) * h(n) = \sum_{k=0}^{n} x(k)h(n-k) \qquad (4.12)$$

One disadvantage with Equation 4.12 is that the multiplication process is done over the entire array of input values, while in practice the impulse response is usually short and most multiplications result in zeros. We can avoid this unnecessary multiplication by using the commutative property of the convolution as shown later in the section.

The Equation 4.13 is the discrete time representation of a first order differential equation

$$y_k = a_0 x_k + b_1 y_{k-1} \qquad (4.13)$$

We can obtain a solution of the above equation through convolution process if the digitized version of the unit impulse response is provided. Assuming the discrete time impulse response is a series $h(0), h(1)...h(n-1), h(n)$ and the input samples are $x(0), x(1)...x(n-1), x(n)$ then the convolution is simply a multiply and add operation as shown in Equation 4.14.

$$y(k) = \sum_{i=0}^{k} h(i)x(k-i) \qquad (4.14)$$

Note: Although the impulse response may be infinite in length, but after a while the response becomes negligible and for all practical purpose values beyond the k_{th} sample is treated as 0.

We can prove that Equation 4.14 is indeed a solution of Equation 4.13 (or Equation 4.1) with the following analogy,

Assuming the input is a unit impulse

$$x(k) = \delta(k) = \begin{cases} 1, k = 0 \\ 0, k \neq 0 \end{cases}$$

Substituting the unit impulse value $x(k)$ into Equation 4.13 we get the following series,

$$y_0 = h_0 = a_0$$
$$y_1 = h_1 = a_0 b_1$$
$$y_2 = h_2 = a_0 b_1^2$$
$$\vdots$$
$$y_n = h_n = a_0 b_1^n$$
(4.15)

But for an arbitrary input signal $x(k)$, we get

$$y_0 = a_0 x(0)$$
$$y_1 = a_0 x(1) + a_0 b_1 x(0)$$
$$y_2 = a_0 x(2) + a_0 b_1 x(1) + a_0 b_1^2 x(0)$$
$$\vdots$$
$$y(k) = a_0 x(k) + a_0 b_1 x(k-1) + ... a_0 b_1^k x(0)$$
$$y(k) = \sum_{i=0}^{k} h(i) x(k-i)$$

We can see that the coefficients of above equations match the impulse response of the Equation 4.13, which is given by Equation 4.15.

Properties of Convolution

The convolution of two different sequences can be combined in different ways,

Commutative Property

The order in which two sequences are convolved is not important. The following equality holds,

$x(n) * h(n) = h(n) * x(n)$

Associative Property

If two systems with responses $h_1(n)$ and $h_2(n)$ are connected in series, an equivalent system is one that has a response equal to the convolution of $h_1(n)$ and $h_2(n)$.

$x(n) * \{ h_1(n) + h_2(n)\} = \{x(n) * h_1(n)\} + \{x(n) * h_2(n)\}$

Distributive Property

If two systems with responses $h_1(n)$ and $h_2(n)$ are connected in parallel, an equivalent system is one that has a response equal to the sum of $h_1(n)$ and $h_2(n)$.

$x(n)*\{ h_1(n) + h_2(n)\}= \{x(n)* h_1(n)\}+ \{x(n)*h_2(n)\}$

Graphical representation of Convolution process

As descried in Equation 4.14 the Convolution is simply a multiply and add process, thus, for any discrete time input sequence *x(k)* and the discrete impulse response sequence *h(k)* of the system, the output sequence *y(k)* may be computed using Equation 4.16,

$$y(k) = \sum_{i=0}^{k} h(i)x(k-i) \qquad (4.16)$$

You may recognize that we only need to perform the multiplication process for the range of numbers in which the impulse response *h(n)* has non zero values. Let's take an example of a system that has the following impulse response,

h(0)=3
h(1)=2
h(2)=1
h(3)=0
.
h(n)=0

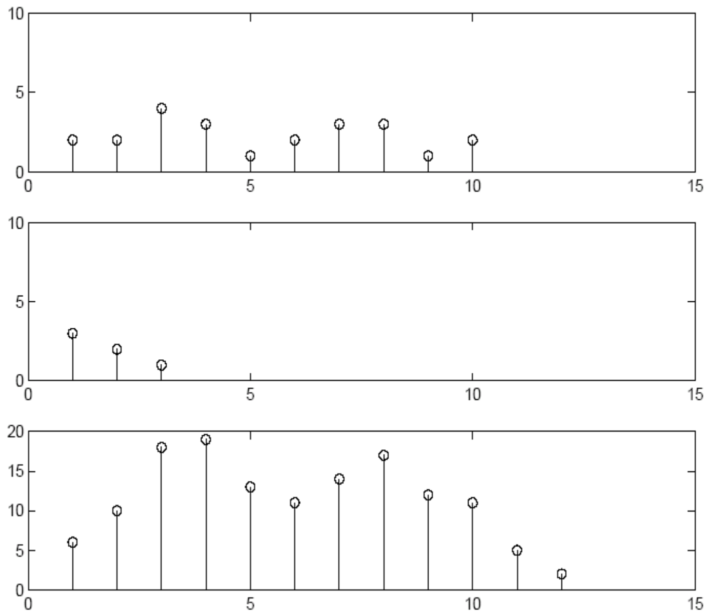
And the input sequence as shown in Figure 4.10.

Notice, the impulse response has zero values beyond the range $k > 2$. Thus, the convolution operation may be reduced as shown in Equation 4.17.

$$y(n) = \sum_{k=0}^{2} h(k) \times x(n-k) \qquad (4.17)$$

To illustrate the steps of discrete convolution, let's say we have input sequence *x(n)* and *h(k)* as shown in Figure 4.10. The out *y(n)* is computed as followings,

$$y(0) = 3\times3 + 2\times0 + 1\times0 = 9$$
$$y(1) = 3\times1 + 2\times3 + 1\times0 = 9$$
$$y(3) = 3\times3 + 2\times2 + 1\times1 = 14$$
$$y(4) = 3\times3 + 2\times3 + 1\times2 = 17$$
$$y(5) = 3\times1 + 2\times3 + 1\times3 = 12$$
$$y(6) = 3\times2 + 2\times1 + 1\times3 = 11$$



****** Insert Figure 4.10 here ******

Figure 4.10.The convolution of discrete time sampling, the arrows showing computation of $y(4) = 3\times3 + 2\times3 + 1\times2 = 17$.

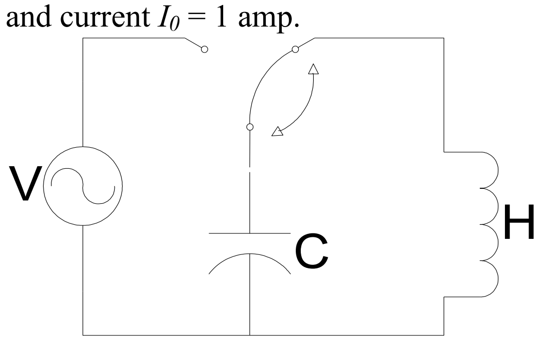
The Convolution operation forms the basis for the digital filtering technique that we will discuss in chapter 6 and 7.

Second order differential equations

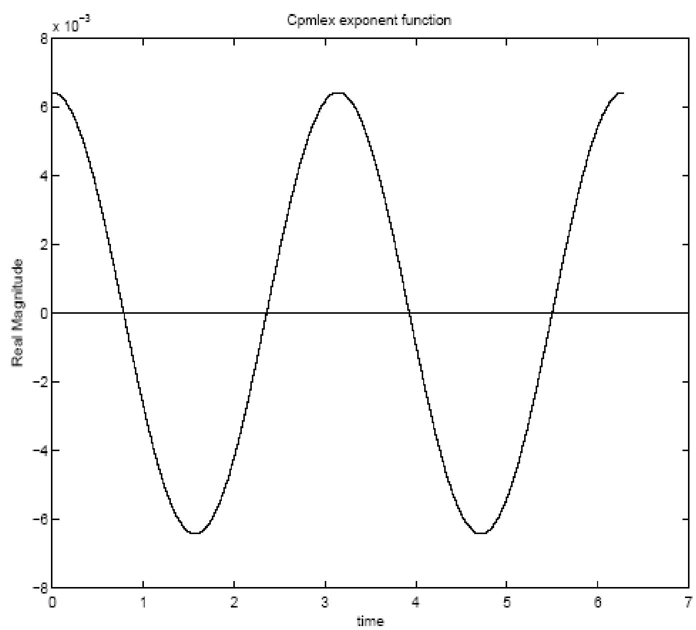
The simple exponential response of the first order differential equation was easy to visualize, but the second order differential equations are more complex in their response; simply, because there are two energy storage elements and their different possible combinations produce varying responses. Before we proceed further with a full mathematical development it would be helpful to create an intuitive feelings about the behavior of such systems in which two energy storage elements are in a loop, such as an inductor and a capacitor, one is capable of storing the current and the other is capable of storing the voltage.

A system as shown in the circuit of Figure 4.11 will serve the purpose for this example. We would like to see the inductive current as the response to the input voltage applied on the capacitor. Let's say the switch S1 on the capacitor *C* was originally connected to the voltage supply, letting the capacitor store certain amount of charge. Once the capacitor was fully saturated we throw the switch towards the inductor *L*, creating a loop between the inductor and the capacitor. The capacitor will start feeding the current to the inductor and the inductor will start building-up the voltage across its terminals. The inductor will gain the energy loss from the capacitor, but then the change of voltage across the inductor will start feeding the charge back into the capacitor. The charge gain by the capacitor back from the inductor will be seen by the inductor as a current source and it will start building the voltage all over again, this back and forth yo-yo of energy loss and gain will last forever as long as we have ideal components.

The rise and fall of the charge on the capacitor and the current on the inductor is a sinusoidal function of time whose amplitude and wavelength depends upon the component values of the inductor and the capacitor of the circuit. To be more precise the frequency of oscillation ω will be exactly equal to the value $\frac{1}{\sqrt{LC}}$ and is shown in the Figure 4.11 for the circuit component of L=1 and C=1 with the initial voltage V_0.=1V and current I_0 = 1 amp.



****** Insert Figure 4.11a here ******

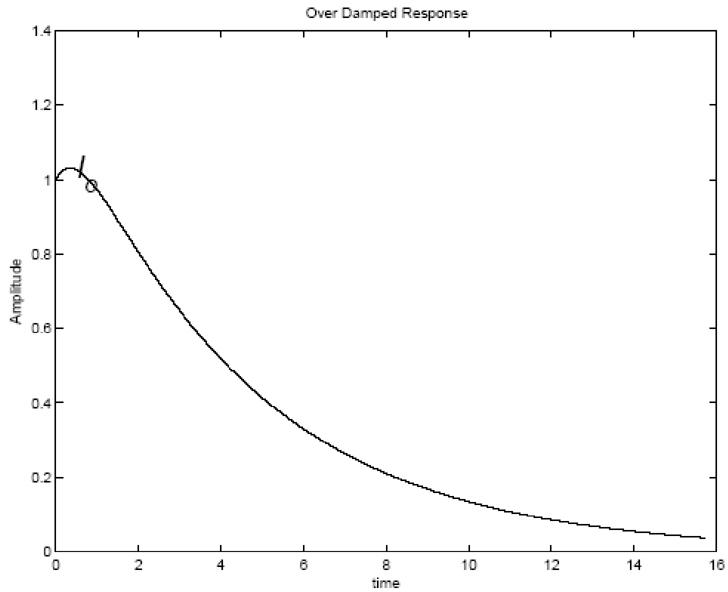


****** Insert Figure 4.11b here ******

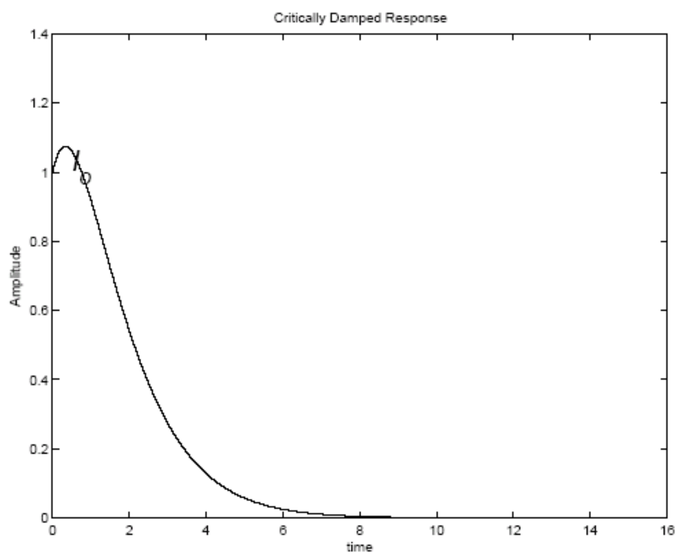
Figure 4.11. A circuit with two energy storage elements creating a sinusoid response

Since, nothing is ideal in this world and there will be a resistance to the current buildup in the inductor and a resistance to the charge build-up in the capacitor there will be a gradual loss of the energy and the sinusoid will die down eventually. We can expedite the loss by simply adding a resistive element to the circuit. This addition of a Resistor will not only add to the exponential loss but we will also see a decrement in the wavelength of the sinusoidal wave. The effect will be seen as if the sinusoid is being sandwiched between the two exponent curves, one rising from the negative and the other falling from the positive, both reaching the datum eventually, while squeezing the sinusoid along the way, see the Figure 4.12.

There is a chance that a fast exponent decay will not let the sinusoid ring at all and the whole thing will die down without showing any up and down motion at all. Otherwise, there will be a gradual decrement in the ringing and finally vanishing in the oblivion as time goes by. The exact phenomenon depends upon two factors, the quantity $\frac{R^2}{4L^2}$ and $\frac{1}{LC}$, if the resistor is placed in series and $\frac{1}{4(RC)^2}$ and $\frac{1}{LC}$ if the resistor is placed in parallel (we will develop the mathematics later in the section, but for now we will only use the terms). If $\frac{R^2}{4L^2}$ is greater then $\frac{1}{LC}$ we will not see any ringing at all, call it an over-damped condition, but if $\frac{R^2}{4L^2}$ is less then $\frac{1}{LC}$ there will be some ringing before reaching the finality, call it an under-damped condition. There is one critical value when $\frac{R^2}{4L^2}$ is just equal to $\frac{1}{LC}$ and this is the transition between being able to see a trough of the wave or not, a critically damped condition. The three responses are presented in the Figure 4.12.a, 4.12.b and 4.12.c. for a series *RLC* circuit and Figure 4.13.a, 4.13.b and 4.13.c. for a parallel *RLC* circuit .



****** Insert Figure 4.12a here ******



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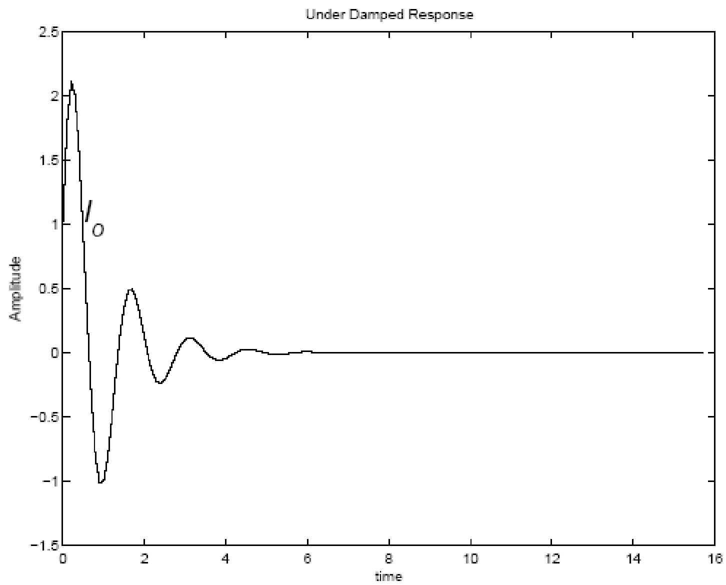
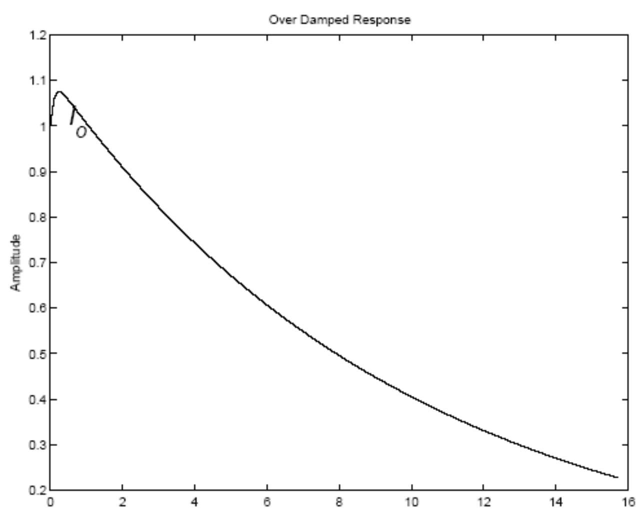


****** Insert Figure 4.12c here ******

Figure 4.12. The series *RLC* circuit of second order differential equations. a) over-damped, b) Under-damped, c) critically damped responses



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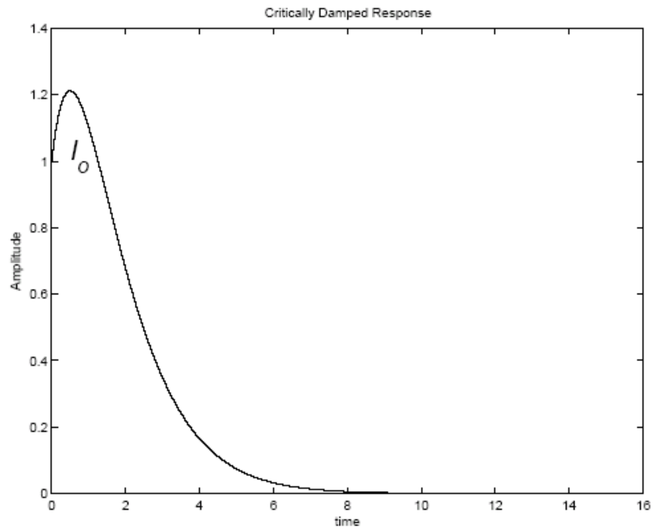

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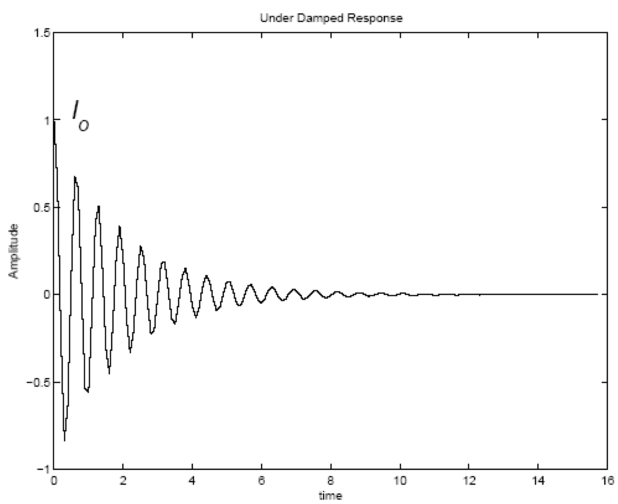


****** Insert Figure 4.13c here ******

Figure 4.13. The parallel *RLC* circuit of second order differential equations. a) Over-damped, b) Under-damped, c) critically damped responses

We have just discussed how inductors and capacitors form a resonant circuit and how adding a resistor puts a damper to the natural frequency. Our primary goal in this section is to study the output (i.e. the current on the inductor) in response to an input, the voltage on the capacitor. In this section we will discuss the networks of electrical components and see how they form a system that effects an input excitation. We will analyze the system of resistors, inductors and the capacitors (*RLC*) in series as well as in parallel combination. The Figure 4.12 is an example of series circuit and Figure 4.13 is the parallel circuit. Our goal is to know how certain input frequencies are attenuated and others pass through without any change, as we will be using them in our quest for designing filters. We begin with the mathematical formulation of the second order differential equation formed as a result of combining the three elements of an *RLC* circuit.

General form of the second order differential equations
The following is the general form of the second order differential equation with two energy storage elements,

$$a_0 y + a_1 \frac{dy}{dt} + a_2 \frac{d^2 y}{dt^2} = x(t) \qquad (4.18)$$

The method of obtaining the solution is the same as that of the first order differential equation. We will derive the natural response of zero input condition giving us the homogenous solution and a forced response providing a particular solution, (the forced response will be due to an external input excitation applied). The complete solution is obtained by adding the homogenous solution and the particular solution.

Natural response

This is the response due to the internal stored energy only. There is no external input force, making *x(t)* equal to zero and the equation 4.18 becomes,

$$a_0 y + a_1 \frac{dy}{dt} + a_2 \frac{d^2 y}{dt^2} = 0$$

Let's work through an example to find the solution, as we did with the first order differential equation. We will go through the series as well as parallel combination of *RLC* network simultaneously. The circuit of Figure 4.12 is a network of a resistor, capacitor and an inductor (*RLC*) in series and 4.13 is the network in parallel. In both cases, the capacitor was charged, initially with the voltage V_0, before being switched to the network.

The followings are the relationship between the current and voltages across different elements in the circuit.

$$i_C = C\frac{dv_C}{dt} \qquad i_R = \frac{v_R}{R} \qquad i_L = \frac{1}{L}\int v_L dt$$

$$v_C = \frac{1}{C}\int idt \qquad v_R = Ri_C \qquad v_L = L\frac{di_L}{dt} \qquad \frac{dv_L}{dt} = L\frac{d^2 i_L}{dt^2}$$

At the time the switch was thrown towards the capacitor, the Kirchoff's current and voltage law describes the relationship for the parallel network as,

$$i_C + i_R + i_L = 0$$

$$v_C = v_R = v_L$$

And for the series network as,

$$v_C + v_R + v_L = 0$$

$$i_C = i_R = i_L$$

We get a second order differential equation as a result of summing the current in case of the parallel network, resulting in the following homogenous equation

$$C\frac{dv_C}{dt} + \frac{v_R}{R} + i_L = 0 \tag{4.19}$$

Substituting the value of $v_R = L\frac{di_L}{dt}$ and $\frac{dv_C}{dt} = L\frac{d^2 i_L}{dt}$ in Equation 4.19 we get

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = 0 \tag{4.20}$$

In case of the series network, summing the voltage provides the following equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = 0 \tag{4.21}$$

Taking the derivative of Equation 4.21 we get,

$$\frac{d^2 i_L}{dt^2} + \frac{R}{L}\frac{di_L}{dt} + \frac{1}{LC}i_L = 0 \tag{4.22}$$

Let's define the following terms,

For series network

$$\alpha = \frac{R}{2L}$$

And for parallel network

$$\alpha = \frac{1}{2RC}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\alpha_d = \sqrt{\alpha^2 - \omega_o^2} \qquad \text{if } \alpha^2 < \omega_o^2$$

$$j\omega_d = j\sqrt{\omega_o^2 - \alpha^2} \qquad \text{if } \alpha^2 > \omega_o^2$$

Assuming the current *i* as an exponential function of time,

$$i_L = Ae^{st}$$

$$\frac{di}{dt} = sAe^{st}$$

$$\frac{d^2 i}{dt^2} = s^2 Ae^{st}$$

Substituting these expression into the homogenous solution of Equation 4.20 and 4.22 we get,

$$s^2 Ae^{st} + s2\alpha Ae^{st} + \omega^2 Ae^{st} = 0 \tag{4.23}$$

The two solutions for *s* are

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \tag{4.24}$$

Both s_1 and s_2 satisfy the Equation 4.23 and we get the following solution describing the current on the inductor as a function of time,

$$i_L = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{4.25}$$

$$i_L = A_1(e^{-\sigma t} \times e^{(\sqrt{\alpha^2-\omega_o^2})t}) + A_2(e^{-\sigma t} \times e^{-(\sqrt{\alpha^2-\omega_o^2})t}) \qquad (4.26)$$

Where value of A_1 and A_2 are to be obtained from the initial conditions.

The value $\alpha = \frac{1}{2RC}$ for a parallel network and $\alpha = \frac{R}{2L}$ for a series network determines the exponential decay, the value $\omega_o = \frac{1}{\sqrt{LC}}$ determines the original frequency of the sinusoid and the value $\omega_d = \sqrt{\alpha^2 - \omega_0^2}$ determines the modified damped frequency of the sinusoid due to the presence of a resistive element in the circuit. The nature of the damped frequency ω_d suggests three possible answers,

a) Roots Real and distinct if $\alpha^2 > \omega_o^2$
b) Roots Real and equal if $\alpha^2 = \omega_o^2$
c) Roots are complex if $\alpha^2 < \omega_o^2$

The two exponents of Equation 4.26 have an implied decay rate that primarily depends upon the value of α. The value $\alpha = \frac{R}{2L}$ is for the resistor in series, and for the resistor in parallel $\alpha = \frac{1}{2RC}$. The quantity under square root $(\sqrt{\alpha^2 - \omega_0^2})$ needs further investigation. It should be noticed that the root is real for an over-damped circuit, since by definition it means $\frac{R^2}{4L^2}$ or $(\frac{1}{4RC^2})$ is greater then $\frac{1}{LC}$. On the other hand if $\frac{R^2}{4L^2}$ is less then $\frac{1}{LC}$ the terms inside the square root $\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ becomes a negative number, an under-damped condition, whose value can only be evaluated by interchanging the terms and multiplying it by the imaginary operator $j = \sqrt{-1}$. The term $j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ or $(j\sqrt{\omega_0^2 - \alpha^2})$ is our new damped frequency of oscillation. The newer frequency $\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ is less then the original $\omega_o = \sqrt{\frac{1}{LC}}$ by factor $\frac{R^2}{4L^2}$. The third option of critically damped condition of $\alpha^2 = \omega_o^2$ merely indicates a transition from over-damped to under-damp condition.

Evaluating the coefficients A1 and A2

The solution presented in the Equation 4.25 requires one to evaluate the coefficients A_1 and A_2 based on the initial conditions (the voltage V_0 present on the capacitor and the current I_0 on the inductor at time $t=0$). It should be noted that for the series combination of the *RLC* circuit the current I_0 gradually builds up in time but at the instance t=0 there is no current, that means I_0 is 0, while in the parallel combination of the *RLC* circuit the least resistive path to the current is through the inductor, resulting in an instantaneous current I_0 on the inductor originating from the capacitor.

Thus, for the series circuit the initial conditions are,

$$i_L = 0 = A_1 + A_2 \qquad t = 0$$

$$\frac{di_L}{dt}(0) = \frac{V_0}{L} = (A_1 s_1 + A_2 s_2) \quad t = 0$$

$$A_1 = \left(\frac{1}{s_1 - s_2}\right)\frac{V_0}{L} \qquad A_2 = \left(\frac{1}{s_2 - s_1}\right)\frac{V_0}{L} \tag{4.27}$$

And for the parallel *RLC*

$$i_L = I_0 = A_1 + A_2 \qquad t = 0$$

$$\frac{di_L}{dt}(0) = \frac{V_0}{L} = (A_1 s_1 + A_2 s_2) \quad t = 0$$

$$A_1 = \left(\frac{1}{s_1 - s_2}\right)\left(\frac{V_0}{L} - s_2 I_0\right) \qquad A_2 = \left(\frac{1}{s_2 - s_1}\right)\left(\frac{V_0}{L} - s_1 I_0\right) \tag{4.28}$$

Roots Real and Distinct

If $\alpha^2 > \omega_o^2$ then the values of $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$ becomes real and the result is the sum of two exponent curves with no ringing of the sinusoid (an over-damped condition). Substituting the value $\omega_d = \sqrt{\alpha^2 - \omega_0^2}$ in s_1 and s_2, the two coefficients for the series *RLC* circuit are reduced to,

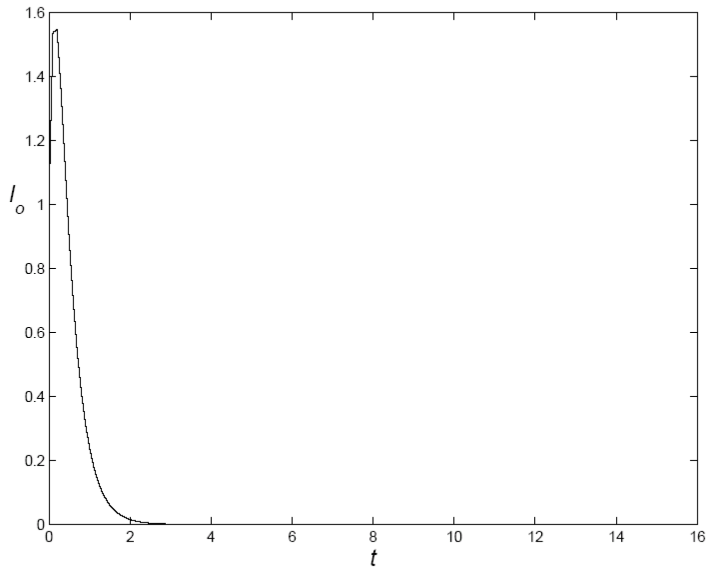
$$A_1 = \frac{V_0}{2L\omega_d} \qquad A_2 = -\frac{V_0}{2L\omega_d}$$

And the current on the inductor is defined as,

$$i_L = \frac{V_0}{2L\omega_d}\left(e^{-\sigma t} \times e^{(\sqrt{\alpha^2 - \omega_o^2})t}\right) - \frac{V_0}{2L\omega_d}\left(e^{-\sigma t} \times e^{-(\sqrt{\alpha^2 - \omega_o^2})t}\right) \tag{4.29}$$

The output i_L of Equation 4.29 is plotted as a function of time in Figure 4.14 for the following component values in the series *RLC* circuit of Figure 4.12,

$R = 1\Omega \qquad C = 0.5F \qquad L = 0.1H \qquad V_0 = 1V$



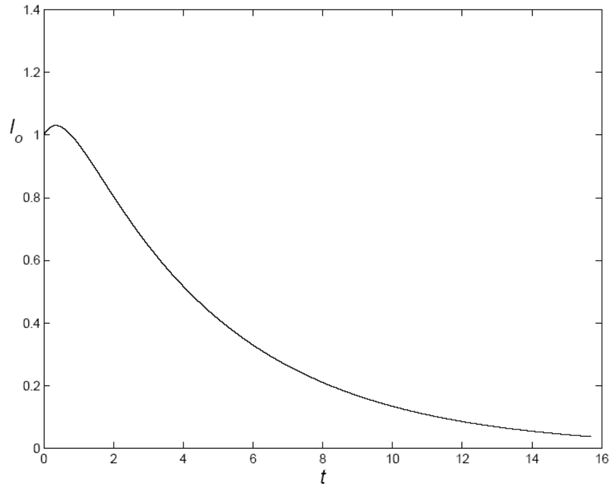
****** Insert Figure 4.14 here ******
Figure 4.14. The exponent curves squeezing the sinusoid, a) over-damped,
b) Under-damped, c) critically damped response of a second order differential equation

Similarly, for the parallel *RLC* network the current on the inductor can be defined by substituting the constants of Equation 4.28 as follows,

$$i_L = \left(\frac{1}{s_1 - s_2}\right)\left(\frac{V_0}{L} - s_2 I_0\right)\left(e^{-\sigma t} \times e^{(\sqrt{\alpha^2 - \omega_o^2})t}\right) + \left(\frac{1}{s_2 - s_1}\right)\left(\frac{V_0}{L} - s_1 I_0\right)\left(e^{-\sigma t} \times e^{-(\sqrt{\alpha^2 - \omega_o^2})t}\right) \tag{4.30}$$

The output i_L of Equation 4.30 is shown in Figure 4.15 as a function of time for the following component values,

$R = 1\Omega$ $C = 1F$ $L = 5H$ $V_0 = 1V$ $I_0 = 1A$



****** Insert Figure 4.15 here ******

Figure 4.3. The exponent curves squeezing the sinusoid, a) over-damped,
b) Under-damped, c) critically damped response of a second order differential equation

As you can see the algebra gets very involved but you can use Matlab to solve the equations. The figures were being drawn using Matlab scripts and the examples are presented in the Appendix A.

Roots Real and Equal

The value of $\alpha^2 = \omega_0^{\ 2}$ is the critically damped condition, which is essentially a borderline condition where the system is just about going to oscillate, but not quite (a critically damped-condition). By substitution one finds that Ate^{kt} is also solution, thus the combined solution is,

$$i_L = (A_1 t + A_2)e^{-\sigma t} \qquad (4.31)$$

Roots real and equal should not be a design consideration as it creates a very unstable circuit. It is only a mathematical probability that the parameters are exactly equal, but it has no engineering significance. The constant A_1 and A_2 of the Equation 4.31 are evaluated for the initial conditions of I_0 and V_0 as follows,

$$I_0 = A_2 \qquad t=0$$

$$\frac{di_L}{dt} = \frac{V_0}{L} = A_1(e^{-\sigma t}) - \alpha(A_1 t + A_2)e^{-\alpha t}$$

$$\frac{V_0}{L} = A_1 - \alpha I_0 \qquad t=0$$

$$A_1 = \frac{V_0}{L} + \alpha I_0$$

$$A_2 = I_0$$

For the parallel *RLC* circuit the current on the inductor i_L is defined as,

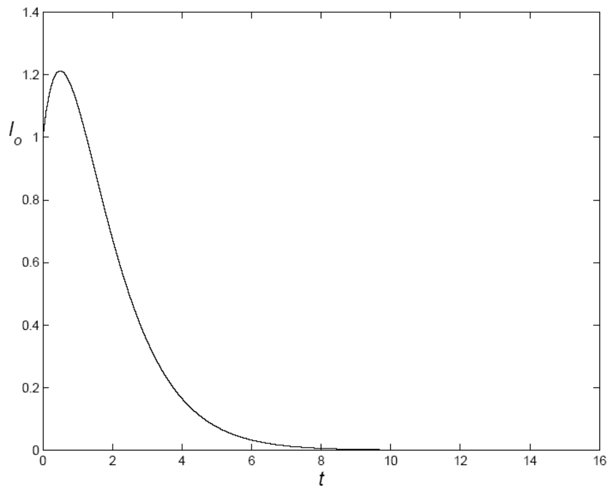
$$i_L = (\frac{V_0}{L} + \alpha I_C)te^{-\alpha t} + I_C e^{-\sigma t} \qquad (4.32)$$

For the series *RLC* circuit $I_c = 0$ and the current is defined as

$$i_L = (\frac{V_0}{L})te^{-\alpha t} \qquad (4.33)$$

The output i_L of Equation 4.32 for the parallel *RLC* circuit is shown in Figure 4.16 as a function of time for the following component values,

$R = 2\Omega \qquad C = 1F \qquad L = 1H \qquad V_0 = 1V \;\; I = 1amp$

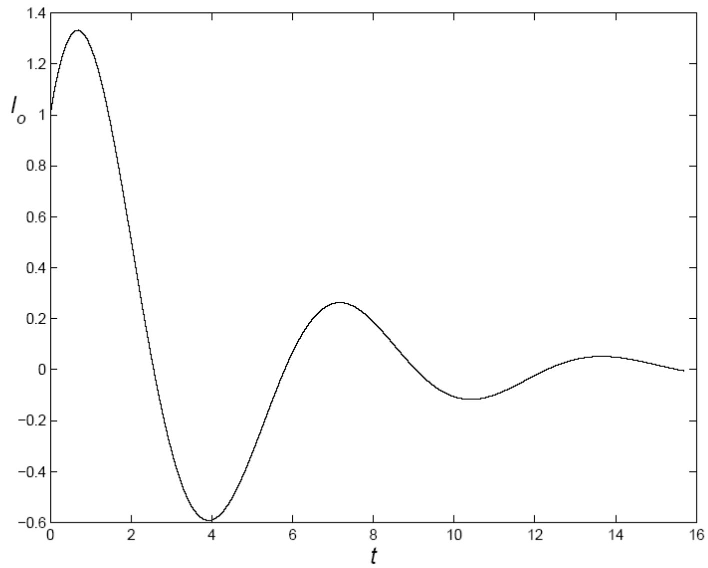


****** Insert Figure 4.16 here ******

Figure 4.16. The output current as a function of time for a parallel circuit critically damped.

The output i_L for series *RLC* circuit of Equation 4.33 is shown in Figure 4.17 as a function of time for the following component values,

$R = 2\Omega \qquad C = 1F \qquad L = 1H \qquad V_0 = 1V \;\; I = 1amp$



****** Insert Figure 4.17 here ******

Figure 4.17. The output current as a function of time for a series critical damped circuit.

Roots Complex

If $\alpha^2 < \omega_o^2$ then the square root becomes a complex number, requiring interchanging the terms ω_o^2 and α^2 by multiplying the root with the operator $j = \sqrt{-1}$. The two exponents give us the following solution,

$$i_L = A_1(e^{-\sigma t} \times e^{j(\sqrt{\omega_o^2-\alpha^2})t}) - A_2(e^{-\sigma t} \times e^{-j(\sqrt{\omega_o^2-\alpha^2})t})$$

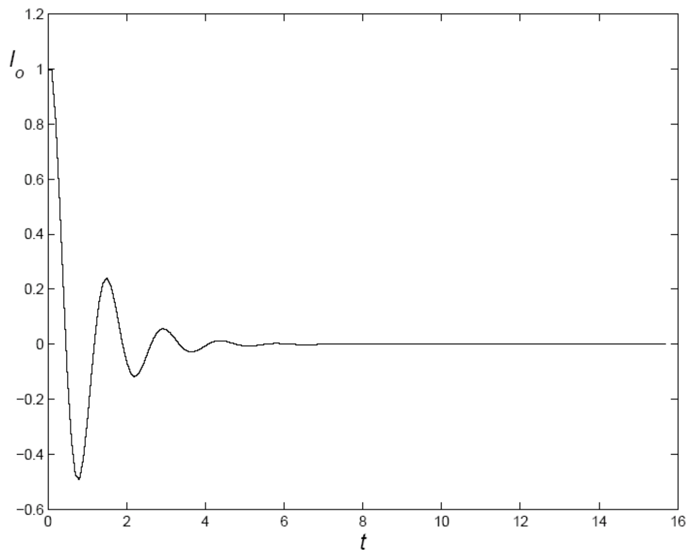
This is the under-damped condition indicating two complex frequency of oscillation $\pm j\sqrt{\omega_o^2 - \alpha^2}$, one rotating in a clockwise direction and the other counter clockwise (see the Figure 2.1 describing two complex conjugate waves). The new wavelength is less then the original frequency ω_0 by a factor of α. With each frequency, there is an exponential decay multiplier $A_1(e^{-\sigma t})$ and $A_2(e^{-\sigma t})$. The system will respond with a sinusoid that will soon die down with an exponential decay rate of $e^{-\alpha t}$.

Using the identity $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$ and substituting the value of A_1 and A_2 as described in the Equation 4.28 for the parallel RLC circuit, we get the following result describing the current on the inductor as a function of time,

$$i_L = \frac{V_0}{\omega_d L} e^{-\alpha t} \sin \omega t + I_0 e^{-\alpha t}(\cos \omega_d t + \frac{\alpha}{\omega_d} \sin \omega_d t) \qquad (4.34)$$

The Figure 4.18 is the plot of the current as a function of time for the following circuit component for the parallel *RLC* circuit of Figure 4.13.

$R = 2\Omega \qquad C = 0.05F \qquad L = 1H \qquad V_0 = 1V$



****** Insert Figure 4.18 here ******

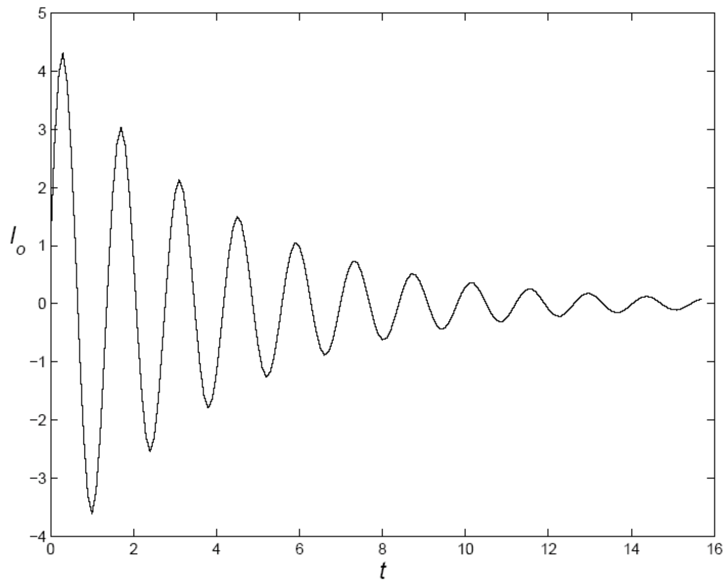
Figure 4.18. The plot of Equation 4.34, indicating the response of an under-damped parallel *RLC* circuit.

For the series *RLC* circuit the initial current I_0=0, and the Equation 4.34 is reduced as follows,

$$i_L = \frac{V_0}{\omega_d L} e^{-\alpha t} \sin \omega t \tag{4.35}$$

The Figure 4.19 is the plot of the current as a function of time for the following circuit component for the series RLC circuit of Figure 4.12.

$R = 2\Omega \qquad C = 1F \qquad L = 0.05H \qquad V_0 = 1V$



****** Insert Figure 4.19 here ******
Figure 4.18. The plot of Equation 4.34, indicating the response of an under-damped series *RLC* circuit.

Forced Excitations

The natural response being discussed in the previous section was obtained for the zero input condition since the input force was removed after the time *t > = 0*. Now we consider the zero state condition where an input force is applied after the time *t >= 0*. We begin with the discussion of the step and impulse response and then derive the response to an arbitrary input using the Convolution process as we did with the first order differential equations.

Step Response

A constant current *I* of a unit magnitude applied to an *RLC* network as shown in Figure 4.13 may be considered a step input *u(t)*. The response to such an excitation should be considered independently for the three conditions namely; roots complex ($\alpha^2 < \omega_o^2$) the under-damped condition, roots real ($\alpha^2 > \omega^2$) the over-damped and roots equal ($\alpha^2 = \omega_o^2$) the critically damped condition. The following analysis use the parallel *RLC* network as shown in the Figure 4.13

Applying the Kirchoff's Current Law to the circuit of Figure 4.13 for the unit step function *u(t)*,

$$\frac{d^2 i_L}{dt^2} + 2\alpha \frac{di_L}{dt} + \omega_0 i_L = u(t) \tag{4.36}$$

The particular solution is obtained from the new input condition of the constant current and we get the following solution

$$i_P(t) = 1 \qquad t \geq 0 \tag{4.37}$$

The general solution is obtained for the three damped conditions of roots real, roots complex and roots equal by adding the homogenous solution to the particular solution

$$i(t) = i_H(t) + i_P(t)$$

Roots real

We get the following general solution by combining the homogenous solution of Equation 4.25 and the particular solution of Equation 4.37.

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + 1$$

In order to resolve the coefficients A_1 and A_2 we need two independent equations, that we can obtain using the following initial conditions

$$i_L(0) = 0 = A_1 + A_2 + 1 \qquad \frac{di_L}{dt}(0) = 0 = A_1 s_1 + A_2 s_2$$

$$A_1 = \frac{s_2}{s_1 - s_2} \qquad A_2 = \frac{-s_1}{s_1 - s_2}$$

The step response is therefore,

$$i(t) = \left[\frac{s_2}{s_1 - s_2} e^{s_1 t} + \frac{-s_1}{s_1 - s_2} e^{s_2 t} + 1 \right] u(t) \tag{4.38}$$

The coefficients s_1 and s_2 are real for the roots real condition,

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

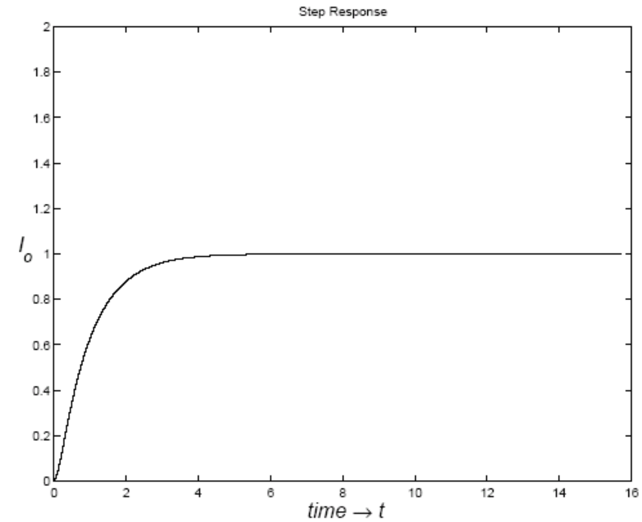
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

For the parallel *RLC* network,

$$\alpha = \frac{1}{2RC} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

The Figure 4.20 is a plot of the Equation 4.38 for the following circuit components of the Figure 4.13.

$R = 0.1\Omega \qquad C = 1F \qquad L = 1H \qquad V_0 = 1V$



****** Insert Figure 4.20 here ******
Figure 4.3. The inductive current of the parallel *RLC* circuit in response to a unit step input.

Roots complex
The under-damped condition follows the same pattern as that of Equation 4.38, except; now the constant s_1 and s_2 are complex conjugate numbers, since $\omega_o^2 > \alpha^2$, At this point we will introduce the phasor equivalent of the complex number s_1 and s_2 to simplify the multiplication operation,

$$s_1 = -\alpha + j\sqrt{\omega_o^2 - \alpha^2} \qquad s_1 = -\alpha + j\omega_d \qquad |s_1| = \sqrt{\alpha^2 + \omega_d^2}$$

$$s_2 = -\alpha - j\sqrt{\omega_o^2 - \alpha^2} \qquad s_2 = -\alpha - j\omega_d \qquad |s_2| = \sqrt{\alpha^2 + \omega_d^2}$$

The polar representation of s_1 is defined as,

$$\frac{\pi}{2} + \phi = \tan^{-1}\frac{j\omega_d}{\alpha}$$ (The operator *j* adds $\frac{\pi}{2}$ to the phase angle*)*

$$\omega_0 = |s_1| = |s_2| = \sqrt{\alpha^2 + \omega_d^2}$$

$$s_1 = \omega_0 e^{j(\frac{\pi}{2}+\phi)} \qquad s_2 = \omega_0 e^{-j(\frac{\pi}{2}+\phi)}$$

Substituting the complex values of s_1 and s_2 into Equation 4.38 we get the following simplified term

$$i(t) = \left[\frac{\omega_o}{s_1 - s_2}\left(e^{-j(\frac{\pi}{2}-\phi)}e^{(-\alpha+j\omega_d)t} - e^{j(\frac{\pi}{2}+\phi)}e^{-(-\alpha+j\omega_d)t}\right) + 1\right]u(t)$$

$$i(t) = \left[\frac{\omega_o e^{-\alpha t}}{s_1 - s_2}\left(e^{(j\omega_d - \frac{\pi}{2} - \phi)t} - e^{-(j\omega_d - \frac{\pi}{2} - \phi)t}\right) + 1\right]u(t)$$

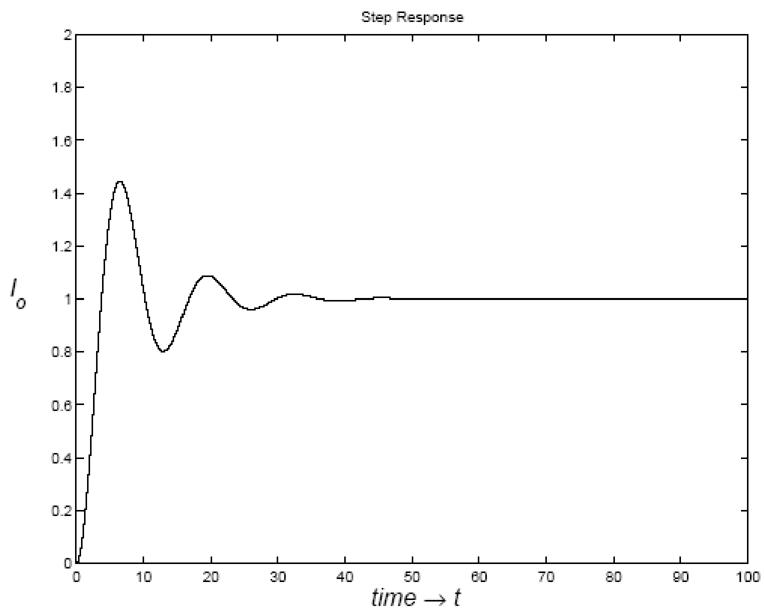
$$i(t) = \left[\frac{\omega_o e^{-\alpha t}}{\omega_d} (\sin(\omega_d - \frac{\pi}{2} - \phi) + 1 \right] u(t)$$

$$i_L(t) = \left[\frac{\omega_o e^{-\alpha t}}{\omega_d} (1 - \cos(\omega_d - \phi) \right] u(t) \qquad (4.39)$$

The graph in Figure 4.21 shows the under-damped frequency response to the unit step function of Equation 4.39, for the following circuit components.

$R = 2\Omega \qquad C = 1F \qquad L = 1H \qquad V_0 = 1V$

It should be noticed that the exponentially decaying sinusoid reaches the value of 1 as time progresses.

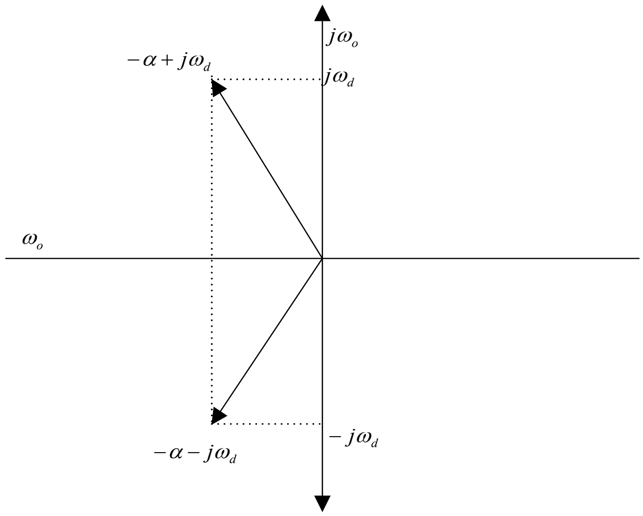


****** Inser Figure 4.21 here ******

Figure 4.21 The step response of complex roots

Complex plane

The complex number representation of the coefficients s_1 and s_2 can best be described as a vector on a rectangular coordinate system. The x-axis is the damping factor α and the y –axis is the damped frequency ω_d , while ω_0 is the vector magnitude. Notice the damping factor α is always on the negative side of the quadrant for positive resistor values. This is the case with physical components in *RLC* circuits. (The concept of negative resistance appears in some networks with feedback amplifiers that act like negative resistors, this is beyond the scope of our analysis.) The Figure 4.22 describes the rotation of such a vector on the complex plane with the magnitude ω_0 and the angle of rotation $\phi = \tan^{-1} \frac{\omega_d}{\alpha}$.



****** Insert Figure 4.22 here ******

Figure 4.3. The vector representation of the complex frequency s_1

Quality factor

The relationship between the damped frequency ω_d and the exponential damping factor α is obvious from the definition $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$. The decrease in ω_d is in proportion to increase in α. The ratio $\frac{\omega_0}{2\alpha}$ can be described as a quality factor in a system with second order differential equation, such as the one being described in the series and parallel *RLC* circuit of Figure 4.12 and Figure 4.13. In order to decrease damping we must decrease α, a zero damping is an infinite *Q*, and that is the case for a true resonant circuit.

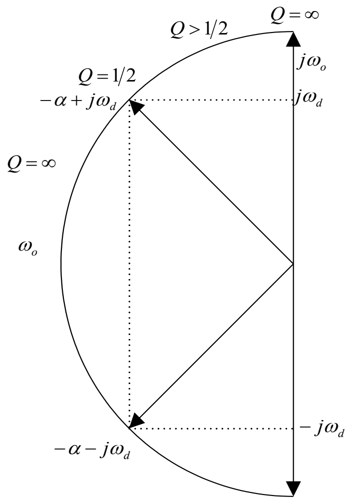
For a series *RLC* circuit the Quality factor *Q* is defined as,

$$Q = \frac{\omega_0}{2\alpha} = \frac{RC}{\sqrt{LC}} = R\sqrt{\frac{C}{L}}$$

And for a parallel circuit,

$$Q = \frac{\omega_0}{2\alpha} = \frac{R}{\sqrt{L/C}}$$

A damped resonant circuit can be described on the basis of the Quality factor *Q* as shown in the Figure 4.23. A $Q < 1/2$ is an over-damped, $Q = 1/2$ is the critical-damped, and $Q > 1/2$ is the under-damped and $Q = \infty$ is a true resonant circuit with no loss of energy.



****** Insert Figure 4.23 here ******

Figure 4.23. The Quality factor representation of the complex number vector $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$ on a rectangular coordinate system

Unit Impulse Response

Finding the response to an impulse function is simply a matter of determining the energy stored during a short impulse, as defined in the previous section of the first order differential Equations. In order to determine the initial conditions, first, we would analyze the affect of the impulse on the capacitor voltage. We know the area under the impulse; it is equal to 1 by definition. The charge injected due to an impulse, will all be consumed by the capacitor, in order to build-up a voltage across the terminals.

$$\int \delta dt = CV = 1$$

$$V = \frac{1}{C}$$

The same voltage will appear across the inductor as the rate of change of the current *i*,

$$V = L\frac{di}{dt} \qquad \text{or} \qquad \frac{di_L}{dt}(0) = \frac{1}{LC} = \omega_0^2 \tag{4.40}$$

The particular solution is given as

$$i_P = \delta \text{ at } t = 0$$

The homogenous solution is given as

$$i_H = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The complete solution is the homogenous solution plus the particular solution

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \delta$$

For all practical purpose the function $\delta = 0$ at time $t > 0$ making the particular solution

$$i_P = 0 \text{ at } t > 0$$

We solve simultaneous equations to obtain the coefficients A_1 and A_2,

$$i(0) = 0 = A_1 e^0 + A_2 e^0 \qquad\qquad A_1 = -A_2$$

$$\frac{di_L}{dt}(0) = \omega_o^2 = A_1 s_1 e^0 + A_2 s_2 e^0$$

$$A_1 = \frac{\omega_o^2}{(s_2 - s_1)} \qquad A_2 = \frac{-\omega_o^2}{(s_2 - s_1)}$$

The current across the inductor for the under-damped case is therefore,

$$i_L(t) = \left[\frac{\omega_0^2}{s_1 - s_2}(e^{s_1 t} - e^{s_2 t})\right]u(t)$$

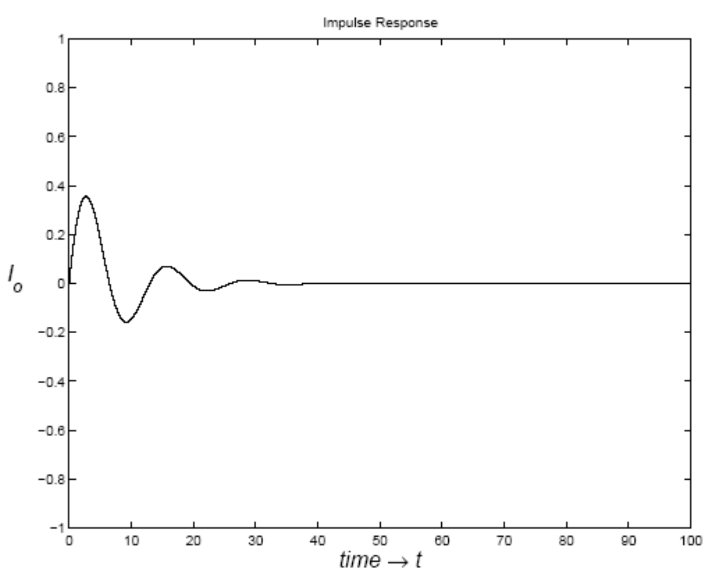
$$i_L(t) = \frac{\omega_o^2}{(2j\omega_d)}(e^{-\alpha t + j\omega_d t} - e^{\alpha t - j\omega_d t})$$

$$i_L(t) = \frac{\omega_o^2}{\omega_d} e^{-\alpha t} \sin \omega_d t \qquad (4.41)$$

The Equation 4.41 must be multiplied with the unit step function *u(t)* in order for it to be valid for all time *t*, which merely indicates that at time $t < 0$ the function is 0.

$$i_L(t) = u(t)\frac{\omega_o^2}{\omega_d} e^{-\alpha t} \sin \omega_d t \qquad (4.42)$$

The Figure 4.24 is the graph of the impulse response. Notice the similarities between natural response and the impulse response.



****** Insert Figure 4.24 here ******

Figure 4.24 The impulse response of the second order differential equation for the Equation 4.42.

Scaled Impulse Response

The response given in the Equation 4.42 is for the unit impulse function whose area under the curve is equal to 1. Any other impulse which is a fraction λ of the unit impulse will produce the scaled response accordingly as defined in the Equation 4.43,

$$i(t) = \lambda u(t) \frac{\omega_o^2}{\omega_d} e^{-\alpha t} \sin \omega_d t \qquad (4.43)$$

Response to an arbitrary input

The convolution process discussed in the previous section of the first order differential equation is also applicable to the second order system. The impulse response convolved with subsequent input to the system is the response to an arbitrary input. The input signal is treated as a series of delayed pulses as shown in the Figure 4.9 and the output is obtained by time-delayed addition of individual responses.

If $h(t - t_k)$ is the delayed impulse response and $i_s(t_k)$ is the current input then the output is computed as the following Convolution integral

$$v(t_n) = \sum_{k=0}^{n} i_s(t_k) h(t - t_k) \Delta \qquad (4.45)$$

We will be using the Convolution method later on when we develop the algorithm to realize the digital filters.

Summary

In this chapter we discussed the input and output relationship of systems such as electrical circuits that are governed by differential equations. We established Convolution as a method of solving the equations and the discrete time Convolution was discussed as a possible solution where a computer may be used for processing the input and output. We analyzed the series and parallel electrical networks for their varying response by solving the corresponding differential equation. The goal in this chapter was to discuss the method of Convolution as a solution of differential equation (it would form the basis of our filter design in the coming chapters) and to see the expected response of systems that are governed by differential equations.