Chapter 3 Fourier Transform

In this chapter

- Periodic function as a complex number function
- Fourier Transform
- FFT algorithm

Introduction

We discussed in Chapter 1 the basics of Fourier analysis and established a mathematical way of describing an arbitrary function by using its frequency components. The hidden message was that an event could be considered as cyclic in nature composed of sinusoidal frequencies, the fundamental and its integral multiples called harmonics. We have found complex numbers as the best way to represent a cyclic function and developed the rules to perform mathematical operations using the Gaussian operator $\sqrt{-1}$. We found that sinusoidal function can be described as complex exponent function by using Euler's identity.

In this chapter we will discuss the transformation of a function from time domain to frequency domain, the culmination of Fourier series into Fourier Transform. The frequency domain gives us another dimension of analyzing events that are extremely difficult in time domain or sometimes not even possible. We presented an example in chapter one where a mysterious output was observed when the input was a periodic function, but with the help of Fourier analysis, the problem was quickly identified. We did not include time in our analysis, and you will see that time is irrelevant when we know that the basic periodic function does not change its shape or form as time goes by.

The aim is to develop software algorithms to achieve the result of the theoretical formulation. But for that we need to modify the Fourier formula in a manner that is suitable for digital computer implementation.

We begin with simplifying our long Fourier series representation of a function into its compact notation of the Euler's exponent form and then deriving an expression that only retains the frequency information to give us a frequency domain equivalent of a time domain function. The mathematical derivative is presented to give a perspective of equivalency and show some facts of the time domain properties that are hidden in the frequency domain.

Periodic function as a complex number function

We have already seen that the complex exponent function as determined by Euler is a composition of sin and cosine function.

$$e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$$
$$e^{-j\vartheta} = \cos\vartheta - j\sin\vartheta$$
$$\cos\vartheta = \frac{e^{j\vartheta} + e^{-j\vartheta}}{2}$$
$$\sin\vartheta = \frac{e^{j\vartheta} - e^{-j\vartheta}}{2j}$$

A periodic function *f (t)* in time is a function of arbitrary amplitude, but has a well-defined period. You can think of any function as periodic as long as you assume that the period is infinite. If you believe a function is not periodic, it may only means that you have not waited long enough to find its period. We have seen how to extract the component frequencies in the previous chapter of Fourier analysis, the long and arduous summation of sin and cosine functions and the DC constant. Next, we'll see how to reduce this long series into a compact exponent notation using Euler's identity. The sample time Fourier series is presented again as a refresher in Equation 3.1.

$$f_k(k\Delta t) = a_0 + \sum_{n=1}^{\infty}(a_n \sin(n2\pi f_0 \Delta t) + b_n \cos(n2\pi f_0 \Delta t))$$

$$a_0 = \frac{1}{T}\sum_{k=0}^{N-1} f_k(k\Delta T) \times \Delta T$$
$$a_n = \frac{2}{T}\sum_{k=0}^{N-1} f_k(k\Delta T)\sin(n2\pi f_0 k\Delta T)\Delta T$$
$$b_n = \frac{2}{T}\sum_{k=0}^{N-1} f_k(k\Delta T)\cos(n2\pi f_0 k\Delta T)\Delta T$$

Where f_0 is the fundamental frequency and nf_0 are the harmonics.

The continuous time equivalent of the discrete time function is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \times \sin(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \times \cos(n\omega_0 t)$$

Where the coefficients are,

$$a_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

(3.1)

Replacing the sin and cosine functions of Equation 3.1 with its equivalent exponent notation and replacing $1/j$ with $-j$, we get,

$$f(t)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{-ja_n}{2} \times (e^{jn\omega_0 t} - e^{-jn\omega_0 t}) + \sum_{n=1}^{\infty} \frac{b_n}{2} \times (e^{jn\omega_0 t} + e^{-jn\omega_0 t})$$

$$= a_0 + \sum_{n=1}^{\infty} (\frac{-ja_n + b_n}{2})e^{jn\omega_0 t} + (\frac{ja_n + b_n}{2})e^{-jn\omega_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} (A_n)e^{jn\omega_0 t} + (B_n)e^{-jn\omega_0 t}$$

(3.2)

$$A_n = \frac{b_n - ja_n}{2}, B_n = \frac{b_n + ja_n}{2}$$

Similar substitution could be made for the Fourier coefficients and the coefficients can be described using the exponent form,

$$A_n = \frac{1}{T}\int_0^T f(t)(e^{-jn\omega_0 t})dt$$

(3.3)

$$B_n = \frac{1}{T}\int_0^T f(t)(e^{jn\omega_0 t})dt$$

$$(3.4)$$

The remaining constant a_0 can also be defined in terms of exponent as shown below

$$A_0 = A_0 e^{j0\omega_0 t} = \frac{1}{T}\int_0^T f(t)e^{0t} \tag{3.5}$$

We can merge the coefficients A_0 into A_n since for n=0 the two exponents are same and simplify the Fourier series some more as shown in Equation 3.6.

$$f(t) = \sum_{n=0}^{\infty}(A_n)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}(B_n)e^{-jn\omega_0 t} \tag{3.6}$$

We could modify the second summation of the Equation 3.6 so that both summation combined would covers the entire spectrum of $n = -\infty$ to $n = +\infty$, but for that we need to redefine the B_n for negative values of n only as follows,

$$B_n = \frac{1}{T}\int_0^T f(t)(e^{-j(-n)\omega_0 t})dt$$

$$B_n = \frac{1}{T}\int_0^T f(t)(e^{jn\omega_0 t})dt$$

$$n = -\infty, \ldots, 0 \tag{3.7}$$

$$f(t) = \sum_{n=-\infty}^{1}(B_n)e^{-jn\omega_0 t} + \sum_{n=0}^{\infty}(A_n)e^{jn\omega_0 t}$$

Comparing the Equation 3.7 and 3.4 we see the two coefficients are equal now, we will give our new coefficient a new name and call it **Fourier Coefficient** $X(n)$ as we have successfully managed to combine the three coefficients of a_0, a_n and b_n into a single form

$$X(n)=\frac{1}{T}\int_{0}^{T} f(t)\times(e^{-jn\omega_0 t})$$
$$n=-\infty,\ldots+\infty \tag{3.8}$$

Combining the two summation for the entire range of the values for $n=-\infty$ to $n=+\infty$ we can describe the Fourier series equivalent of the original function into the complex exponent form as

$$f(t)=\sum_{n=-\infty}^{\infty} X(n)e^{jn\omega_0 t} \tag{3.9}$$

Where the exponent function is the Euler's identity,

$$e^{jn\omega t}=\cos(n\omega t)+j\sin(n\omega t)$$

With the similar analogy we could derive the discrete version of the Fourier coefficient called **Discrete Fourir Transform** into its condensed form

$$X(n)=\sum_{k=0}^{N-1} f(k\Delta t)\times(e^{-jn2\pi f_0 k\Delta t})$$
$$n=0,1,\ldots.,N-1 \tag{3.10}$$

From the Discreter Fourier Transform we could bring back the sampled function as follows

$$f(n)=\sum_{k=0}^{N-1} X(n)\times(e^{jn2\pi f_0 k\Delta t})$$

(3.11)

You may say that now there is a certain amount of elegance in the representation of the Fourier series in Equation 3.11 and 3.9, that we have taken a long series and reduced it to a compact looking function. But we certainly could not use this formula for actual calculations. We need the coefficients a_n and b_n and for that we still have to go through the same computational process that we did with the long series. Hidden in the coefficient of $X(n)$ are a_0, a_n and b_n and the $e^{jn\omega t}$ hides the $\cos(n\omega t)+\sin(n\omega t)$ functions. Still, the Equation 3.11 expresses the relationship between an input and output.

Removing the periodic dependency

One drawback in all Fourier analysis is that it is being assumed that the complex wave has a predefined period. The formula requires you to have data for at least one complete cycle and expects the behavior to be repeated. But who knows what the wave looked like before and what will it look after ward or is it really a periodic wave per se? After all, nature is too complex to have a repeated pattern. And what are the consequences of taking few samples and basing a judgment on the entire function? To answer these questions we really need a formula that does not depend upon the function being a periodic wave and this is what **Fourier Transform** is all about, it is based upon the following principle, a sample from a lot tells you something about a lot and more samples from the lot tells you more about the lot, so let's modify the Euler's formula and remove the periodic dependency and see the consequences of it on the original function.

Euler's equation and non-periodic wave

The Fourier series in its original form is not very practical as is. It requires tremendous amount of calculations and it assumes that the input function is periodic in nature. The Euler's formula or the complex number representation of the Fourier series of Equation 3.11 has hidden the conditionality of wave being periodic in its coefficient $X(n)$, but now we have a dilemma. We cannot bring back the original function, as it does not give us the actual coefficient a_0, a_n and b_n that we need. In this section we will discover the method that will modify the Euler's formula and enables us to deal with any kind of wave, periodic or non-periodic.

To begin our discussion let's assume all functions are periodic in nature. Well, what if we assume every function has a period and its value is infinite, can't argue with that. You would not live long enough to prove otherwise. But according to

Fourier series, to get complete spectrum of the frequencies in a function we need to observe the function for one complete cycle. Does it mean the job can never be completed? Having a period that approaches infinity has other problems too. It would make the fundamental harmonics near 0 as shown in the Equation 3.12.

$$\Delta f = \frac{1}{T_{\to\infty}} \to \frac{1}{\infty} \to 0 \tag{3.12}$$

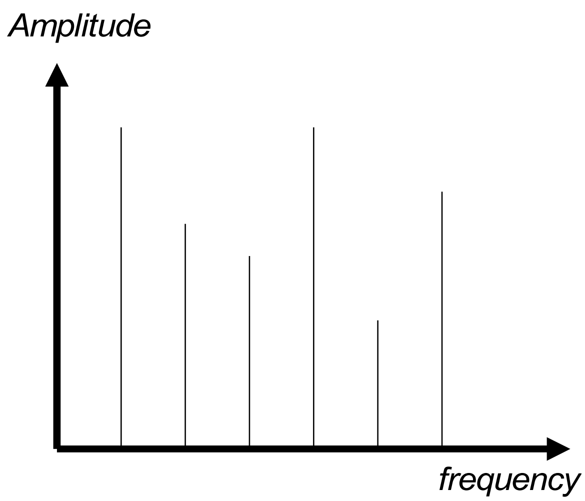
The more sample you take in one-second duration the more you expand your frequency spectrum. Every sample contributes to some degree towards the whole even a single sample, and brings you closer in defining the frequency components of the original function whose period is now assumed to be infinite.

Fourier Transform

The argument presented in the previous section leads us to one conclusion that the factor $\frac{1}{T}$ must be removed from our calculations in order for the Euler's formula to have any practical value. If the wave takes infinite time to advance through one period then its frequency approaches 0. What happens to the spectrum of the frequency, as the wave passes through only the smallest fraction of its period in one second? The spectrum shows only the integral multiple of the fundamental frequency, so as the frequency approaches 0, the interval between frequencies on the spectrum grows narrower and narrower. Eventually as the period approaches infinity the spectrum becomes continuum and all frequencies are known and there is no gap between the frequencies. Since, now every sample is considered a contributing factor towards the frequency component and the number of samples is related to the number of frequencies in our spectrum, it is no longer a time domain any more. The domain is our frequency spectrum and the rate of sampling is Δf instead of Δt.

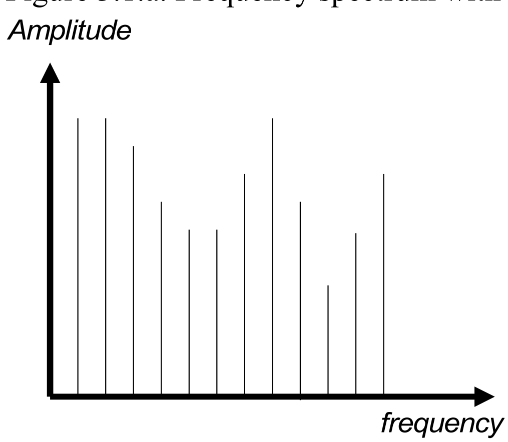
$$\Delta f = \frac{1}{T}$$

The graph of Figure 3.1.a, b and c shows how rate of sampling affects the frequency spectrum. The gap between frequency Δf approaches 0 as the number of samples in one-second approaches infinity.



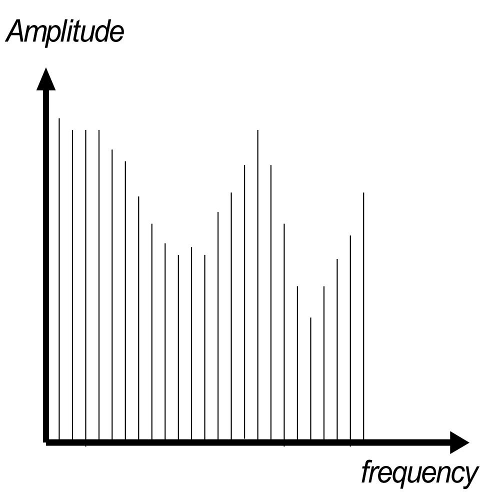
****** Insert Figure 3.1.a here ******

Figure 3.1.a. Frequency spectrum with 12 samples/sec



****** Insert Figure 3.1.b here ******

Figure 3.1.b. Frequency spectrum with 20 samples/sec



****** Insert Figure 3.1.c here ******

Figure 3.1.c. Frequency spectrum with 50 samples/sec

Having realized that frequency a continuum function, the fundamental and the harmonics nf_0 are simply a continuous frequency variable f with no gaps among them. Let's rewrite the Equation 3.10 by shifting the summation halfway negative and replacing $\frac{1}{T}$ with df (as $T \to \infty$) and $n\omega$ with $n2\pi f$, $X(n)$ with $X(\omega)$. We would also like to use the term *g(t)* instead of *f(t),* as *f* is being used for frequency.

$$X(\omega) = df \times \int_{n=-\frac{T}{2}}^{\frac{T}{2}} g(t) \times e^{-j2\pi ft} \times dt \tag{3.13}$$

Where *g(t)* is from Equation 3.9,

$$g(t) = \sum_{n=-\infty}^{\infty} X(\omega) \times (e^{j2\pi ft}) \tag{3.14}$$

We have also switched to dt instead of ΔT in Equation 3.13, since our period *T* is infinite so a small portion of *t* is equal to a small portion of *T.*

Replacing the value of $X(\omega)$ in Equation 3.14 with the Equation 3.13.

$$g(t) = \sum_{n=-\infty}^{\infty} \left\{ df \times \int_{n=-\frac{T}{2}}^{\frac{T}{2}} g(t) \times e^{-j2\pi ft} \times dt \right\} (e^{j2\pi ft}) \tag{3.15}$$

The sticky point in the Equation 3.15 is the multiplying factor df, which is very close to 0, so we will move it out

$$g(t) = \sum_{n=-\infty}^{\infty} \left\{ \int_{n=-\infty}^{\infty} g(t) \times e^{-j2\pi ft} \times dt \right\} (e^{j2\pi ft}) \times df \tag{3.16}$$

All we have done in Equation 3.16 is taken the Equation 3.15 and moved the quantity df to the end. We have also replaced the quantity $\frac{T}{2}$ with ∞, since T is infinite so $\frac{T}{2}$ is also infinite.

The term in the bracket of Equation 3.16 is our **Fourier Transform**. It has the desired property that it does not depend upon information regarding the period. It may be confusing to see that we still have to do the summation all the way up to infinity, but wait till next section and we will have an explanation.

We give the term in the bracket a new name $X(\omega)$ for our Fourier Transform.

$$X(\omega) = \int_{n=-\infty}^{\infty} g(t) \times e^{-j2\pi ft} \times dt \qquad (3.17)$$

And to bring back our original function we have **Inverse Fourier Transform** by substituting the value $G(f)$ back in Equation 3.16.

$$g(t) = \sum_{n=-\infty}^{\infty} X(\omega) \times (e^{j2\pi ft}) \times df \qquad (3.18)$$

But just what does Fourier Transform of Equation 3.17 represent? It is not the actual frequency component. The actual frequency component is our $X(n)$. Let's compare the two.

$$X(n) = \frac{1}{T}\int_{0}^{T} f(t) \times (e^{-jn\omega_0 t})$$

$$n = -\infty, \ldots + \infty$$

$$X(\omega) = \int_{n=-\infty}^{\infty} g(t) \times e^{-j2\pi ft} \times dt$$

(3.19)

The two quantities are very much alike. But the $X(n)$ is the true area divided by the length as given by the term $\frac{1}{T}$, whereas $X(\omega)$ is a relative term. It may not show the actual frequency but it is unique for every frequency in the function. And this is good enough for our analysis.

We can take a complex wave and identify its component objects with Fourier Transform of Equation of 3.17 (we can not call them frequencies as explained above) and with Inverse Fourier Transform the same component objects combined return our original function as in Equation 3.18.

The Fourier Transform gives us a comparative analysis of component frequencies whereas Fourier series give us the true analysis of component frequencies. The Fourier series has Fourier coefficients to get back to the original function and in Fourier Transform the transform itself is being used to bring back the original function.

We have Fourier series in an acceptable form in Equation 3.19 in the form of Fourier Transform. The integral in 3.19 is an improper integral since the domain of integration is an unbound interval. The convergence or divergence of the integral depends entirely on the function $g(t)$, since the magnitude of the term $e^{-j2\pi ft}$ never exceeds 1.

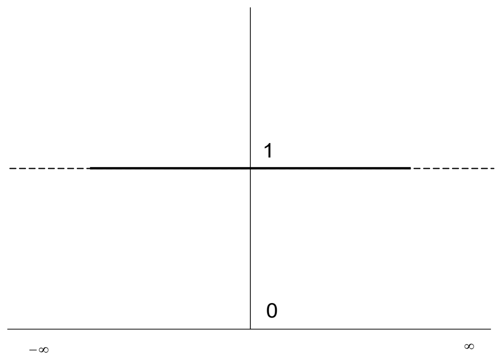
$$\left|e^{-j2\pi ft)}\right| = \left|\cos(2\pi ft) - j\sin(2\pi ft)\right| = (\cos^2(2\pi ft) + \sin^2(2\pi ft)) = 1$$

The Fourier Transform of a function shows us the component frequencies present in any function, but how close we get in approximating the true contents of a function is a topic of discussion next.

You will see the benefit of Fourier Transform from Digital Signal Processing point of view later in the section but a little bit more rigor in mathematics is presented next to show you a brilliant conclusion of Fourier Transform the Heisenberg Uncertainty Principal.

The difference between a true function and the estimation with the Fourier Transform

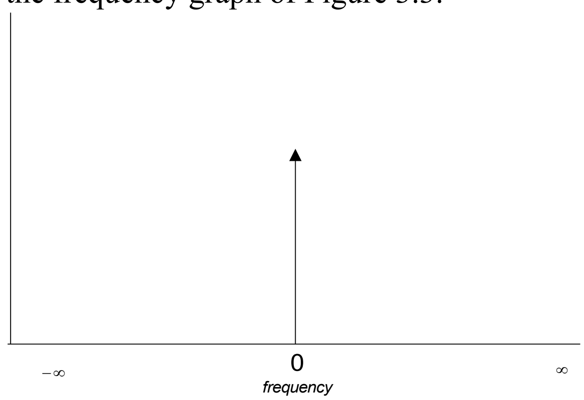
Let's analyze the result for a moment if we have a wave that has an infinite period and find the Fourier Transform of it. The function $g(t) = 1$ is one example that does not oscillate. It goes from –ve to +ve infinity without touching the ground. It has only one frequency $f = 0$.



****** Insert Figure 3.2 here ******

Fig .3.2 A function with a constant value of 1.

If we apply the Fourier Transform and analyze the spectrum it should look like the frequency graph of Figure 3.3.



****** Insert Figure 3.3 here ******

Fig. 3.3 Frequency spectrum of an ideal function with a constant value

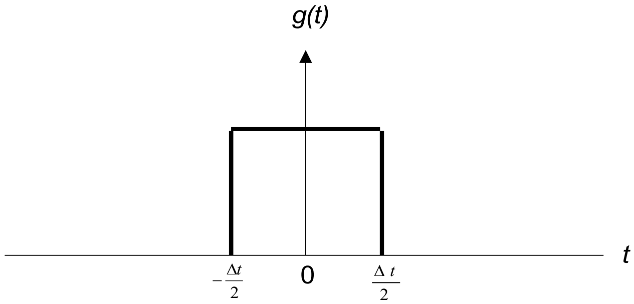
It would be next to impossible to integrate this function over a period as the period stretches all the way up to infinity, so we take a finite time period of ΔT and try to integrate the function. We have to assume that beyond ΔT everywhere the function is 0.

The Examples 3.1 and 3.2 will elaborate the effect of limiting our samples while measuring an event.

Example 3.1

Consider a function whose value is equal to a constant c inside a domain $-t$ and t. The value is 0 outside the domain, a) find the Fourier Transform of the function, b) find the Fourier Transform of an impulse function with unit height, and a domain approaching 0.

$$f(x) = \begin{vmatrix} c for |x| \leq t \\ 0 for |x| \geq -t \end{vmatrix}$$



****** Insert Figure 3.4 here ******

Figure 3.4 Function for example 3.1

a)

According to Equation 3.19

$$X(\omega) = \int_{t=-t}^{t} c \times e^{-j2\pi ft} \times dt = \left[\frac{e^{-j2\pi ft}}{-j2\pi f} \right]_{-t}^{t} \tag{3.19}$$

$$X(\omega) = \frac{c}{\pi f} \left[\frac{e^{j2\pi ft} - e^{-j2\pi ft}}{2j} \right] = c \times \frac{\sin(\pi ft)}{\pi f} \tag{3.20}$$

b)

The unit impulse function is similar to the function being defined in Example 3.1 and shown in figure 3.5. Substituting the value of *c=1* and a small duration δ in Equation 3.20 we get the following transformation.

$$X(\omega) = \frac{1}{\pi f} \left[\frac{e^{j2\pi f \frac{\delta t}{2}} - e^{-j2\pi f \frac{dt}{2}}}{2j} \right] = \frac{\sin(\pi f \frac{\delta t}{2})}{\pi f \frac{\delta t}{2}} \times \frac{\delta t}{2}$$

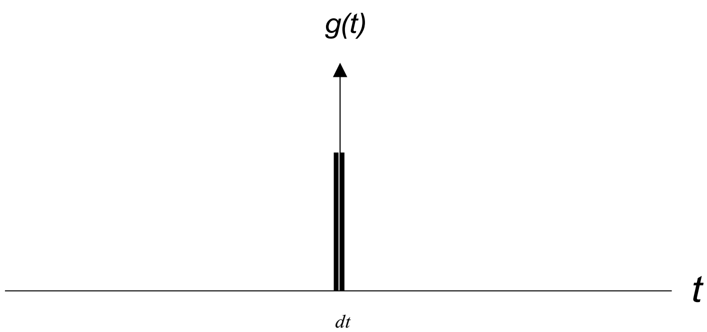
$$\sin(\pi f \frac{\delta t}{2}) = \sin(\frac{\theta}{2})$$

For a very small angle

$$\frac{\sin(\theta)}{\theta} = 1$$

$$X(\omega) = \frac{\sin(\theta)}{\theta} \times \delta t = \delta t_{\theta \to 0}$$

(The unit impulse function commonly known as **Dirac delta function** will be used in extracting a specific term of a sequence when we discuss convolution operation later on in next chapter.)

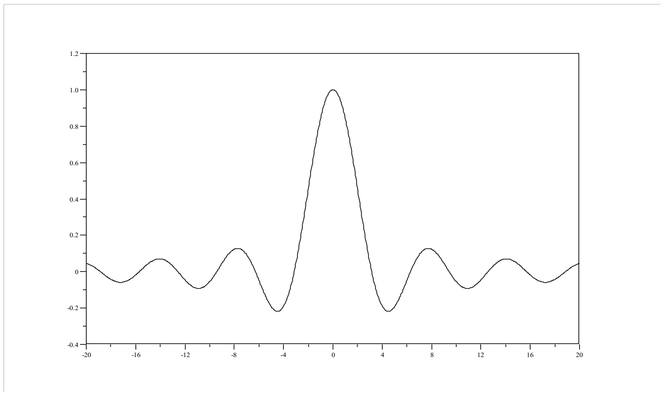


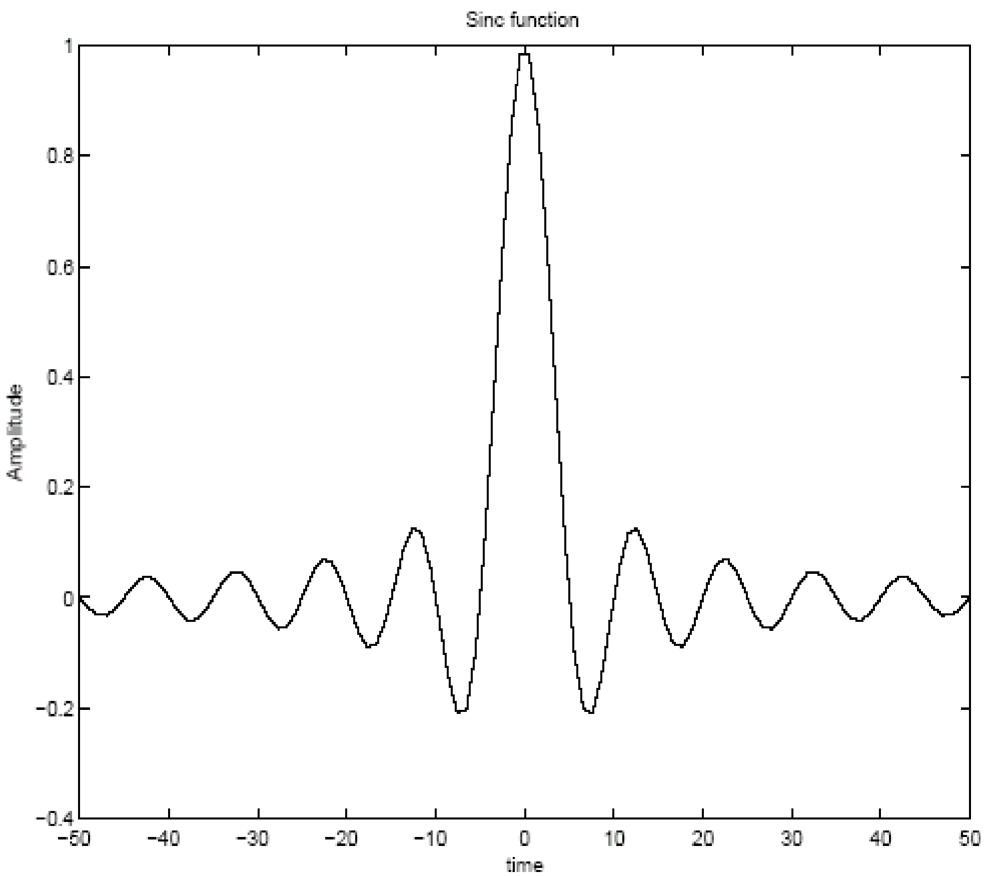
****** Insert Figure 3.5 here ******
Figure 3.5 Unit impulse function

Example 3.2:

Plot the frequency graph of Fourier Transform of Fig 3.1, a) for $t = 0.2$, b) for $t=2$, c) for $t=4$ seconds. The amplitude is being determined as 1 in all three cases.

a)
Plugging the value $c = 1, t = 0.2$ in the Equation 3.2, Figure 3.1 showing the plot of the function $G(f) = \dfrac{\sin(0.2\pi f)}{\pi f}$



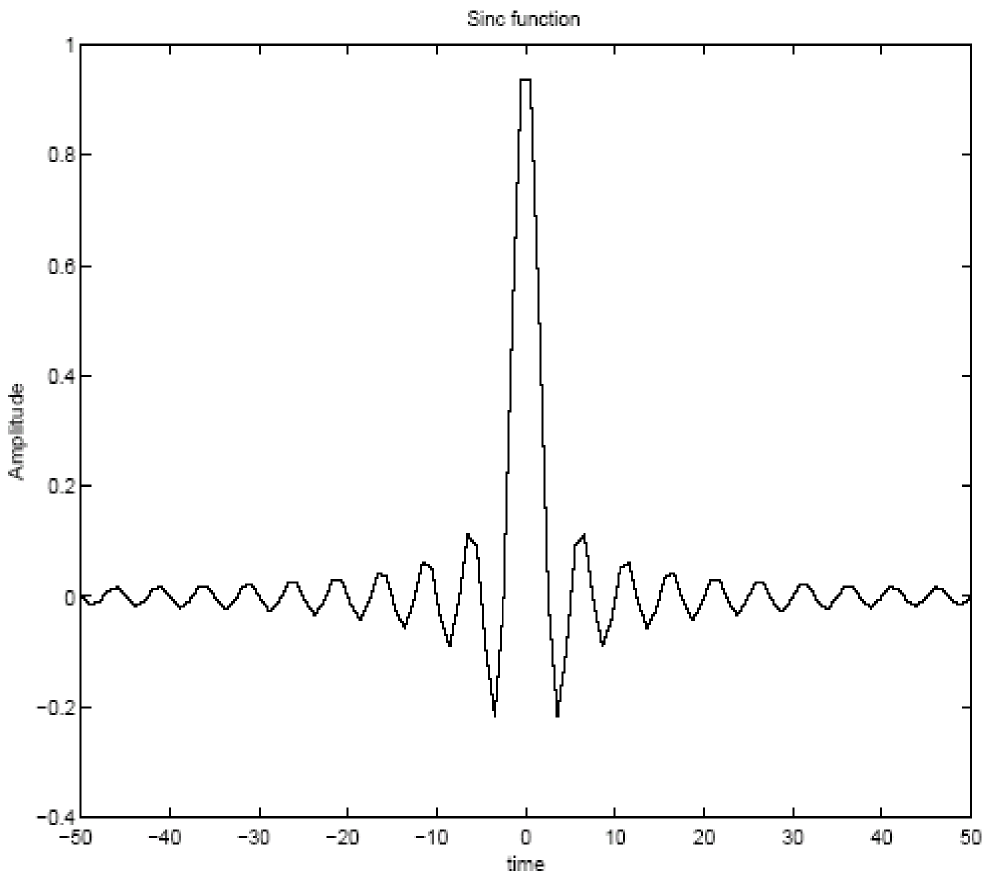


****** Insert Figure 3.6.a here *****

Figure 3.5 a. Graph of the function $X(\omega) = \dfrac{\sin(0.2\pi f)}{\pi f}$

b)

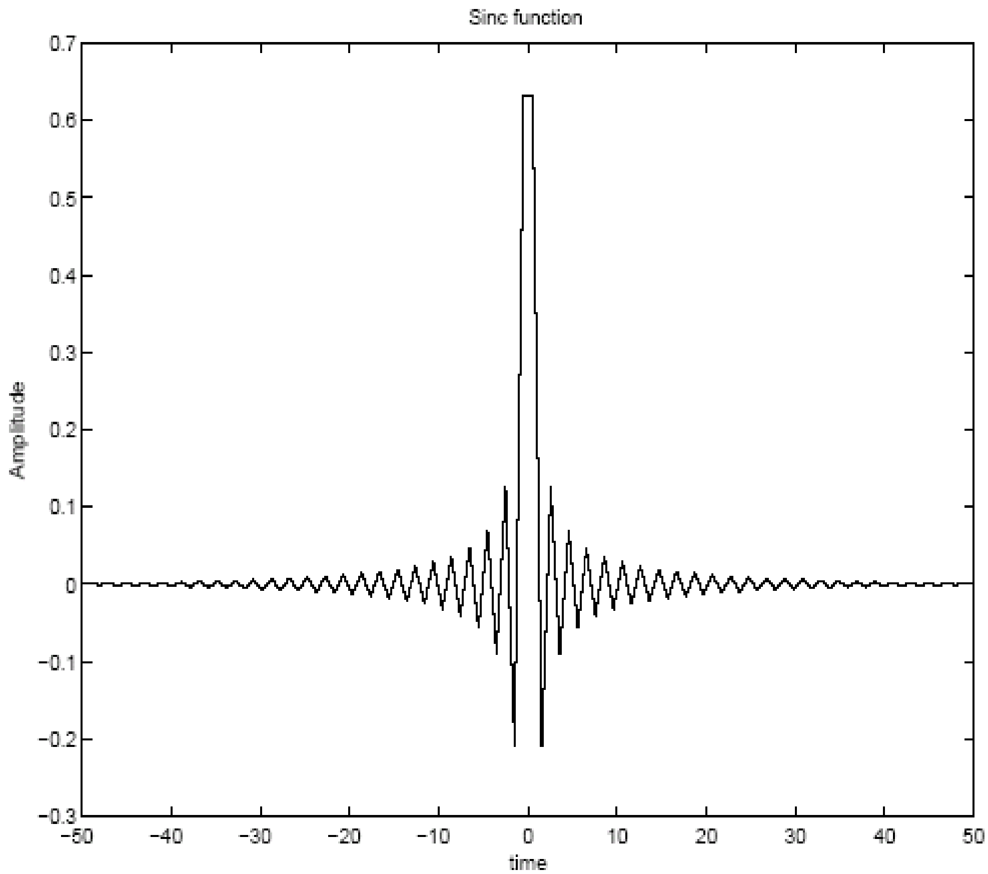
Plugging the value $c = 1, t = 2$ in the Equation 3.2, Figure 3.1 showing the plot of the function $X(\omega) = \dfrac{\sin(2\pi f)}{\pi f}$



****** Insert Figure 3.6.b here *****

Figure 3.6.b. Graph of the function $X(\omega) = \dfrac{\sin(2\pi f)}{\pi f}$

c)
Plugging the value $c = 1, t = 4$ in the Equation 3.2, Figure 3.1 showing the plot of the function $X(\omega) = \dfrac{\sin(4\pi f)}{\pi f}$



****** Insert Figure 3.6.c here *****

Figure 3.6 c. Graph of the function $X(\omega) = \frac{\sin(4\pi f)}{\pi f}$

The Example 3.1 a, b and have an interesting conclusion. The amplitude of the true frequency increases as we increase our period of sample. (Assuming the function is a constant with an infinite period and there is only one frequency with a value of 0 in the function, see Equation 3.12). We should be seeing only one component in our frequency spectrum and it should have the highest value at 0 frequency as in Figure 3.5. But the result in Figure 3.6.a, b and c shows a trend towards true frequency of 0 but there are other frequencies also in our spectrum. In order to achieve the true result of 0 frequency, the duration of sample ΔT should have been infinite.

It is true in real life also, that the more we observe the better we understand. Every new sample brings us closer to a new reality about the function. And in order to achieve perfection we should keep observing forever and that is not possible, so we never achieve perfection. Every time we stop to consolidate our gain we add a little unknown in the formation of the function. Just like Fourier

Transform there is a little bit of uncertainty in our estimation of a true function. Though the uncertainty improves as the time goes by, but it remains with us till the end of time and the moment we stop observing, that we must, we add a little uncertainty and that would show in the Fourier Transform of the function.

If Δt is a measure of sample in time then Δf could be a measure of uncertainty in our sample. The Fourier Transform tells us that the quantity $\Delta t \times \Delta f$ is a constant and in order to reduce the gap in frequencies (reduce the Δf) we must increase our time of sample Δt. There is a relationship of variance or the rate of change of sample in time and the variance in its Fourier Transform, which is given by Werner Heisenberg as the **uncertainty principle.**

$$Var\{h(t)\} \times Var\{H(t)\} \geq \frac{1}{16\pi^2}$$

Discrete Time Fourier Transform

Now, we turn our attention to the real subject matter of implementing digital signal processing, by that, we mean processing of an event to suit a desired outcome by using a computer. In the context of digital signal processing, where a function is sampled in discrete time by a process control system using an A/D converter, the data can be described as an instantaneous summation of each individual frequency component at that particular instance of time. The processing involves integrating, differentiating, smoothing and filtering of analog signals as well as acquiring data for displaying and archive purpose. We only need to go through an elaborate mathematical computations such as Fourier Transform if there is a processing involved that requires extracting a set of frequencies from the input signal or transforming the input signal to a modified form or performing operations such as correlations.

The analog world may be continuous in time but when a computer is being used to process a signal, there is always a finite amount of time between acquisition and subsequent processing, no matter how fast the computer is. The method is essentially a sampling of event taken at discrete time. The mathematics has to be modified slightly to take into account that only at specific time the discrete time coincides with the real time. But it does not pose much problem. We only need two subsequent samples and time duration between each sample to identify a particular frequency as explained by the Nyquist theorem of chapter 1.

If Δt is our sampling period then the *kth* sample time that coincide with the real time is $t = k\Delta t$. Each discrete sample in time of a function *g(t)* is being placed in the perspective *kth* position of the function and is being represented as *g(k)*. If ω is the angular velocity in radians per second then $\omega\Delta t$ is the advancement of the wave in radians per sample.

If *N* is the number of samples per second, there are *N/2* frequencies in the given period and $1/\Delta tN$ will be the period of the fundamental frequency. That gives us the nth frequency component $f_n = n/_{\Delta tN}$, where *n =0,1,2,3,4…N/2*

Modifying the Fourier Transform of Equation 3.16 for discrete time events we have discrete time Fourier Transform in Equation 3.21.

$$G(n/_{\Delta tN}) = \sum_{k=0}^{N-1} \Delta t \times g(k\Delta t) \times e^{-j2\pi\Delta tk\frac{n}{\Delta tN}} \qquad (3.21)$$

Since Δt is our choice, we could normalize the Equation 3.21 by choosing $\Delta t = 1$ and simplify the Fourier coefficient formula as,

$$G(n/_{N}) = \sum_{k=0}^{N-1} g(k) \times e^{-j2\pi k\frac{n}{N}}$$

$$e^{-j2\pi k\frac{n}{N}} = \cos(2\pi k\frac{n}{N}) - j\sin(2\pi k\frac{n}{N})$$

The only two things you need define before the processing is the sampling rate Δt, and the number of samples *N* in the processing and the Equation 3.21 will compute all the harmonics from 0 to N/2. The following is a simple loop program to highlight the number of steps involved,

```
#define N 16;
#define DeltaT 1/16;
Float Rsum,Jsum,freq[N/2];
Float Samples[N];
.
.
.
main()
{
  for (n=0;n<N/2;n++)
  {
     Sum=0;
```

```
      for(k=0;k<N-1;j++)
      {
         RSum=RSum+DeltaT*Samples[k]*cos(2.0*3.14159*k*n/N);
         JSum=JSum+DeltaT*Samples[k]*sin(2.0*3.14159*k*n/N);
      }
      freq[n]=sqrt((Rsum*RSum)+JSum**JSum));
   }
}
```

It is obvious that the program is computationally intensive as is. Just to give you a perspective in real life the number of computations involved, let's take an example of analyzing a voice spectrum on digitized data of sound pattern.
As a first step, we need to determine the maximum frequency that needs to be extracted from the sampled data. The human voice can easily reach up to 4000 Hz. That means we need at least 8000 data point per second. We have n=4000, N=8000 and $\Delta t = {}^{1}\!/_{8000}$.

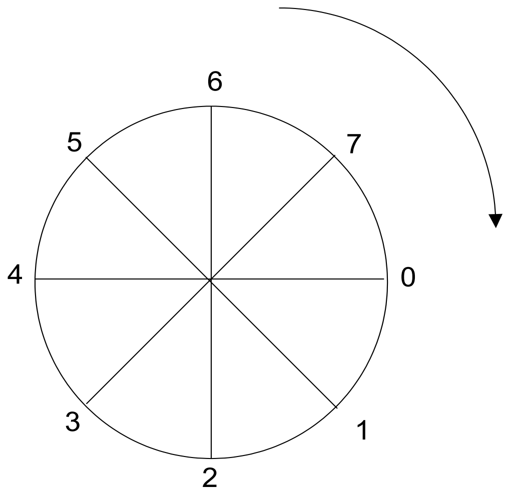
The number of computations is 2x4000x8000 multiplications just to get the magnitude of the Fourier Transform, a very impractical suggestion. Fortunately there are ways to improve the algorithm and that is being discussed next in Fast Fourier Transform.

Fast Fourier Transform

Reducing the number of computations in Discrete Fourier Transform has been a dream of many and if you take a closer look at the algorithm of Listing 3.1, you will see that there are a lot of redundant calculations. It would become clear when we rewrite the Equation 3.21 in a slightly different format,

$$G({}^{n}\!/_{N}) = \Delta t \times \sum_{k=0}^{N-1} g(k) \times e^{-j\frac{2\pi}{N} \times nk} \tag{3.22}$$

The exponent term $e^{-j\frac{2\pi}{N}}$ in the Equation 3.22 is a complex number indicating a point on a circle. The fact that it is multiplied by *nk* shows the point is being rotated around the circle *nk* times. The *–j* only signifies that the rotation in clockwise direction. Let's assume that you are computing Fourier Transform of 8 samples, *N = 8,* hence the number of frequencies *n=4* and you step through each frequency k=0 through 7 times. That means the complex point is rotated *nk=32* times around the circle. But after the first 8 rotations the point simply moves around the circle and repeat its pattern as shown in Figure 3.7.



****** Insert Figure 3.7 here ******

Figure 3.6 A point $e^{-j\frac{2\pi}{N}}$ moves around the circle when multiplied by *nk*.

For every new wave n=0,1,2,3 the multiplication steps are repeated for k=0,1,2,3,4,5,6,7, but notice the point never leaves the circle. In essence $e^{-j\frac{\pi}{4}\times 0}$ is equal to $e^{-j\frac{\pi}{4}\times 8}$ and $e^{-j\frac{\pi}{4}\times 1}$ is equal to $e^{-j\frac{\pi}{4}\times 9}$ etc. Table 3.1 shows the actual complex number vector involved once redundancies are being removed. The column headings are the numerical points and the row headings are the frequencies.

Table 3.1. Complex numbers to be multiplied in an 8 point DFT.

	g(0)	g(1)	g(2)	g(3)	g(4)	g(5)	g(6)	g(7)
G(0/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$
G(1/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 1}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 3}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 5}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 7}$
G(2/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 6}$
G(3/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 3}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 1}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 7}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 5}$
G(4/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$
G(5/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 5}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 7}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 1}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 3}$
G(6/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 2}$
G(7/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 7}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 5}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 03}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 1}$

Look carefully in the Table 3.1 and you will see that every even point has a top and bottom half that are identical. It means we only need to compute the top half and use the computations for the bottom half. This is one big savings in multiplication operation. Now there is a different set of rule, if k=0,1,2,3,4,5,6,7 then 2k are the even points 0,2,4,6 and 2k+1 are the odd points 1,3,5,7.

We can rewrite the Equation 3.22 to reflect the new set of operations as shown in Equation 3.23.

$$G(n/N) = \Delta t \times \{\{\sum_{k=0}^{N/2-1} g(2k)\times e^{-j\frac{2\pi}{N}\times n(2k)}\} + \{\sum_{k=0}^{N/2-1} g(2k+1)\times e^{-j\frac{2\pi}{N}\times n(2k+1)}\}\} \tag{3.23}$$

Let's split the group into odd and even operations and rearrange the table to reflect the new set of computations as shown in Table 3.2.

Table 3.2. Complex numbers to be multiplied in an 8 point DFT with blank entries indicating redundant operations..

	g(0)	g(2)	g(4)	g(6)	g(1)	g(3)	g(5)	g(7)
G(0/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$
G(1/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 1}$	$e^{-j\frac{\pi}{4}\times 3}$	$e^{-j\frac{\pi}{4}\times 5}$	$e^{-j\frac{\pi}{4}\times 7}$
G(2/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 6}$
G(3/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 3}$	$e^{-j\frac{\pi}{4}\times 1}$	$e^{-j\frac{\pi}{4}\times 7}$	$e^{-j\frac{\pi}{4}\times 5}$
G(4/8)					$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 4}$
G(5/8)					$e^{-j\frac{\pi}{4}\times 5}$	$e^{-j\frac{\pi}{4}\times 7}$	$e^{-j\frac{\pi}{4}\times 1}$	$e^{-j\frac{\pi}{4}\times 3}$
G(6/8)					$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 2}$
G(7/8)					$e^{-j\frac{\pi}{4}\times 7}$	$e^{-j\frac{\pi}{4}\times 5}$	$e^{-j\frac{\pi}{4}\times 03}$	$e^{-j\frac{\pi}{4}\times 1}$

We have one more trick of mathematics that we can apply on Equation 3.23. The odd complex number multiplier (2k+1) can be converted into an even complex number by separating the exponent as follows,

$$e^{-j\frac{\pi}{N}(2k+1)} = e^{-j\frac{\pi}{N}n(2k)} \times e^{-j\frac{\pi}{N}n} \tag{3.24}$$

What we have done essentially is rotated the vector that makes the point fall on to the next even location. The operation will become clear when we discuss the vector rotation in the next section, but for now just see the advantage of converting an odd complex number operation into an even complex number operation. We can apply the same divide and conquer rule that we did in the first part of Table 3.2 where even numbers top and bottom halves became identical.

The new equation is

$$G(n/N) = \Delta t \times \{\{\sum_{k=0}^{N/2-1} g(2k) \times e^{-j\frac{2\pi}{N}\times n(2k)}\} + e^{-j\frac{2\pi}{N}\times n} \times \{\sum_{k=0}^{N/2-1} g(2k+1) \times e^{-j\frac{2\pi}{N}\times n(2k)}\}\}$$

If we expand the operation on the right hand side of the equation into a table as we did in for the original computation in Table 3.2, we will see the same repeated pattern emerging in tabulation. No entries are made in the bottom half of the Table 3.3 since the bottom half is identical to the top half. But don't forget that at the end of all multiplication and summation we need to perform one last operation of multiplication with $e^{-j\frac{\pi}{N}n}$.

$$e^{-j\frac{2\pi}{N}\times n} \times \{\sum_{k=0}^{N/2-1} g(2k+1) \times e^{-j\frac{2\pi}{N}\times n(2k)}\}\}$$

k=0,1,2,3
n=0,1,2,3,4,5,6,7

Table 3.3. The complex number multiplier in odd points calculations

	g(1)	g(3)	g(5)	g(7)
G(0/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$
G(1/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 6}$
G(2/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$
G(3/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 2}$

Let's divide and conquer some more, since we are doing so well. Look at every alternate point in the Table 3.3. The top two and the bottom two are same now just like in the Table 3.1. We can eliminate the redundant calculations and reduce computations some more as in Table 3.4.

Table 3.4. The complex number multiplier in odd points calculations, blank entries indicating redundant operation

	g(1)	G(5)	g(3)	g(7)
G(0/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 0}$
G(1/8)	$e^{-j\frac{\pi}{4}\times 0}$	$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 2}$	$e^{-j\frac{\pi}{4}\times 6}$
G(2/8)			$e^{-j\frac{\pi}{4}\times 4}$	$e^{-j\frac{\pi}{4}\times 4}$

G(3/8)			$e^{-j\frac{\pi}{4}\times 6}$	$e^{-j\frac{\pi}{4}\times 2}$

We keep doing rotation and eliminate the redundant computations until we have only one point left.
Let's make some simple substitutions to make the expressions little cleaner and remove the redundancies while expanding the Fourier Transform computations as shown in Equation 3.25.

$$g(2k) = p(k)$$
$$g(2k+1) = q(k)$$
$$e^{-j\frac{\pi}{N}} = W$$

$$G(n/N) = \Delta t \times \{\{\sum_{k=0}^{N/2-1} p(k)\times W^{n(2k)}\} + W^n \times \{\sum_{k=0}^{N/2-1} q(k)\times W^{n(2k)}\}\}$$

$$G(n/N) = \Delta t \times \{$$
$$\{\sum_{k=0}^{N/4-1} p(2k)\times W^{'n(2k)}\} + W^{'n} \times \{\sum_{k=0}^{N/4-1} p(2k+1)\times W^{'n(2k)}\}$$
$$+ W^n \{\sum_{k=0}^{N/4-1} p(2k)\times W^{'n(2k)}\} + W^{'n} \times \{\sum_{k=0}^{N/4-1} q(2k+1)\times W^{'n(2k)}\}\} \tag{3.25}$$

Let's see the total savings we achieve by rotating the point and eliminating the redundant computations,

1st step)
$a(0), a(1) \rightarrow 4$
2nd step)
$W^{'n} \times (b(0), b(1)) \rightarrow 4+2$
3rd step)
$W^n \times (c(0), c(1)) \rightarrow 4+2$
4th step)
$W^n[W^{'n} \times (d(0), d(1))] \rightarrow 4+2+2$

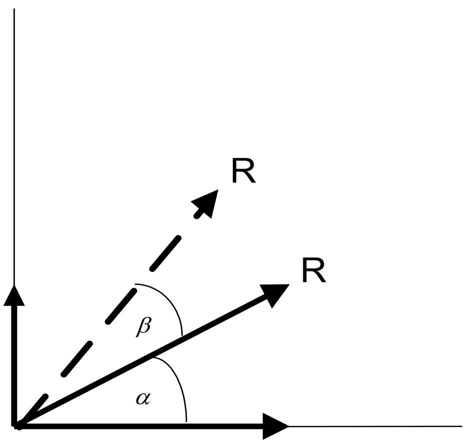
Total = 24

The original Discrete Fourier Transform required 8 x 8 = 64 multiplications that were reduced to only 24 multiplications in the Fast Fourier Transform algorithms. It is a savings of a magnitude in time.

The multiplication operation with $e^{nk\frac{\pi}{N}}$ can be explained as an operation of a vector rotation.

Vector Rotation

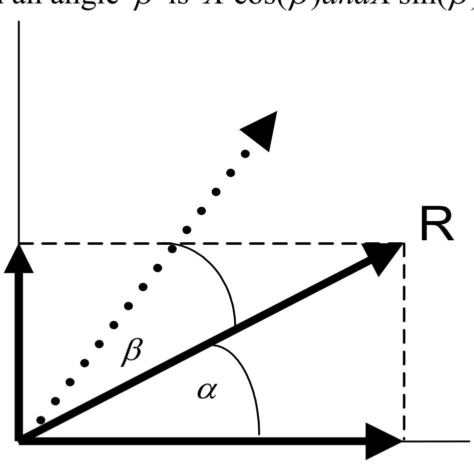
A complex number $e^{-j\alpha}$ is a vector in a Cartesian coordinate system with a magnitude **R** and a phase angle α. Multiplying this vector with another complex number $e^{-j\beta}$ is like rotating through an angle β as shown in Figure 3.8



****** Insert Figure 3.8 here ******

Figure 3.8. Rotation of the vector of magnitude **R** through an angle β

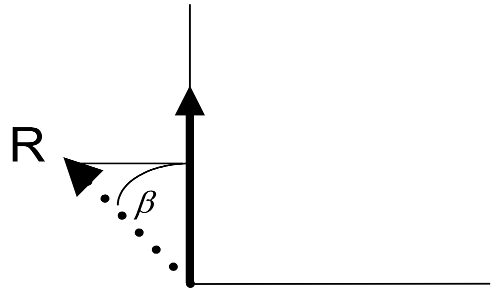
The X component of the vector $X = R\cos(\alpha)$ and the rotation of the X component through an angle β is $X\cos(\beta) and X\sin(\beta)$ as shown in Figure 3.9.a.



****** Insert Figure 3.9.a here ******

Figure 3.9.a. The rotation of X component through angle β

The Y component of the vector $Y = R\sin(\alpha)$ and the rotation of the Y component through an angle β is $Y\cos(\beta) and Y\sin(\beta)$ as shown in Figure 3.9.b.



****** Insert Figure 3.9.b here ******

Figure 3.9.b. The rotation of Y component through angle β

The rotated vector X and Y components are computed as,

$$Y_{rot} = Y\cos(\beta) + X\sin(\beta)$$

$$X_{rot} = X\cos(\beta) - Y\sin(\beta)$$

FFT algorithm

The best way to implement an FFT algorithm is to use the native code of the CPU instruction set, as there is an enormous amount of computation involved. But there are public domain software, such as FFTW (claimed to be the fastest Fourier Transform in the World), Scilab and Grace with built-in FFT routines that you can freely download and use. You will see an implementation of FFT using 'Grace' algorithm in Chapter 6 when we discuss Digital Filters.

Summary

In this chapter, we have taken the long and cumbersome Fourier series and derived a short and concise form using Euler's identity. Although the coefficients of Fourier Transform no longer represent the amplitude of true sine and cosine functions, but from practical point of it is sufficient to use them as relative magnitude of frequencies. We also developed the method Fast Fourier Transform that simplifies the number of computations in Fourier Transform calculations.

The Fourier series in Chapter 1 gave us the amplitude of the component frequencies that were supposed to be infinite in numbers to be a true representative of a function, but the frequencies were discrete integral multiple of fundamental frequency, we have corresponding Discrete Fourier Transform when we use number of samples as oppose to time as the independent variable. The Fourier Transform on the other hand gave us frequencies that were continuous with no gap among them and we have Discrete Time Fourier Transform when we use sampling frequency as the independent variable as shown in the Table 3.5.

Time Duration		
Finite	Infinite	
Fourier Series (FS) $X(k)=\frac{1}{T}\int_0^T x(t)e^{-j\omega_k t}dt$ $k=-\infty,....,+\infty$	Fourier Transform (FT) $X(\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ $\omega\in(-\infty,+\infty)$	Continuous time t
Discrete Fourier Transform (DFT) $X(k)=\sum_{n=0}^{N-1} x(n)e^{-j\omega_k n}$ $k=0,1,....,N-1$	Discrete Time Fourier Transform (DTFT) $X(\omega)=\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$ $\omega\in(-\pi,+\pi)$	Discrete time t
Discrete frequencies, ω	Continuous frequencies, ω	