Chapter 2 Complex number arithmetic

In this chapter

- Complex number representation
- Complex numbers in polar coordinates
- Exponent *e* and the power functions
- The phasor method

Introduction

We established the fact in the previous chapter of Fourier analysis that an arbitrary function could be described using its constituent frequencies in terms of sin and cos functions. The mathematics of the sinusoid was developed using an analogy as if it was a spot on a bicycle wheel moving in time. We could also consider the amplitude as if it was a point on a circle. We needed two numbers to describe the point, in a Cartesian coordinate system the point P(x, y) is the distance in the x-direction and the distance in the y-direction, while in Polar coordinate system it was $r_n \cos(\theta)$ and $r_n \sin(\theta)$. But despite the fact that they were real numbers, we could not perform normal algebraic operations of additions and multiplications upon them. Point (x, y) in essence is a single number and is treated as one entity. One obvious reason of not being able to perform arithmetic is that the algebra does not allow us to have a comma in parenthesis, so mathematician had to invent a different numbering system, just like they did for the negative numbers, real numbers, logarithmic numbers, and for representing points in a coordinate system they invented complex numbers.

In this chapter, you would see how the new numbering system of complex numbers allows us to apply algebraic rules when such numbers are placed in algebraic equations. We would develop the mathematical foundation and establish the rules of arithmetic operations involving the complex numbers that would not only help us analyze the problems in electrical circuits (the same electrical circuits that we intend to simulate using digital signal processing), but also lay down the foundations for solving a broad range of problems that the digital signal processing is intended to solve. We intend to perform filtering of frequencies, besides integration, differentiation and smoothing of sampled data obtained form the analog world.

Complex number representation

If you know your Pythagoras theorem well you will have no difficulty in understanding the complex number arithmetic. As mentioned before, a complex number is nothing but a point on a coordinate system made up of two numbers, essentially, the relative magnitude of the distance in x and y direction from the origin, whereas the absolute distance is computed as the square root of the x squared and the y squared, (according to the Pythagoras theorem of the right angle triangle).

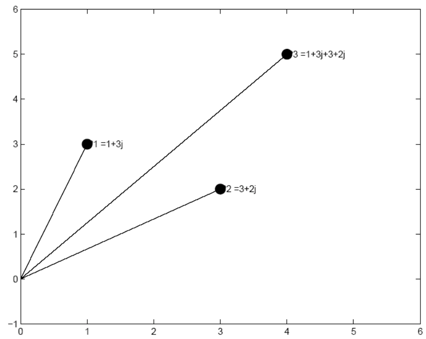

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Figure 2.1. Points in Cartesian coordinate system

Suppose you have two points P_1 (1,3) and P_2 (3,2) and you would like to add them together, the result is the third point P3 (4,5) as shown in the Figure 2.1.

$$(1,3) + (3,2) = (4,5) \qquad (2.1)$$

The Equation 2.1 is not algebraically correct. Commas are not allowed in parenthesis, but then just how do you represent two numbers as distinct as x and y and be algebraically correct? Gauss gave the answer, (Carl Friedrich Gauss 1777-1855 Brunswick, Germany). He introduced the concept of an **imaginary operator** $\sqrt{-1}$. If you multiply the second number with $\sqrt{-1}$ all pieces of the puzzle fall into places. Now you can treat the x and y components of a complex number as if they were two ordinary real numbers as shown below,

$$(1 + \sqrt{-1}(3)) + (3 + \sqrt{-1}(2)) = (4, \sqrt{-1}(5))$$

It is customary to show the coordinate system y-axis as the imaginary axis and x-axis as the real axis.

The imaginary operator $\sqrt{-1}$

The Gaussian operator $\sqrt{-1}$ helps us perform the vector arithmetic using simple algebraic rules. Multiply the y component of the complex number with $\sqrt{-1}$ and all arithmetic operation can be carried out as ordinary algebraic quantities, as shown below,

$$(a + \sqrt{-1}c) \times (b + \sqrt{-1}d) = (ab - cd) + \sqrt{-1}(bc + ad)$$

$$(a + \sqrt{-1}c) + (b + \sqrt{-1}d) = (a + b) + \sqrt{-1}(c + d)$$

$$(a + \sqrt{-1}c) - (b + \sqrt{-1}d) = (a - b) + \sqrt{-1}(c - d)$$

$$\frac{(a + \sqrt{-1}c)}{(b + \sqrt{-1}d)} = \frac{(a + \sqrt{-1}c)}{(b + \sqrt{-1}d)} \times \frac{(b - \sqrt{-1}d)}{(b - \sqrt{-1}d)} = \frac{(ab + cd) + \sqrt{-1}(bc - ad)}{(b^2 + d^2)}$$

The operator $\sqrt{-1}$ (usually written as ***j*** or ***i*** in an equation) follows the rule of multiplication.

$$j = \sqrt{-1}, j^2 = -1, j^3 = -\sqrt{-1}, j^4 = 1, j^5 = \sqrt{-1}.....$$

Complex Conjugate

If $z = x + jy$ is an arbitrary complex number then $z = x - jy$ is the mirror image on the y-axis. The number $z = x - jy$ is the complex conjugate of $z = x + jy$ as shown in Figure 2.2b.

The complex conjugate is usually denoted with an asterisk

$$z* = x - jy$$

The conjugate pair of complex numbers has the following property,

$$\frac{1}{2}(z + z*) = \frac{1}{2}\left[(x + jy) + (x - jy)\right] = x = \text{Re}(z)$$

$$\frac{1}{2}(z - z*) = \frac{1}{2}\left[(x + jy) - (x - jy)\right] = jy = j\,\text{Im}(z)$$

The magnitude of the complex number is obtained by multiplying it with its complex conjugate,

$$zz* = \left[(x + jy) \times (x - jy)\right] = (x^2 + y^2)$$

Thus,

$$|4 + j3| = \sqrt{(4 + j3) \times (4 - j3)} = \sqrt{(4)^2 + (3)^2} = 5$$

To add the two points (3+2j) and (4+5j) add the real values and the imaginary values separately. The result is a new complex number shown graphically in the Figure 2.2

$$(3 + 2j) + (4 + 5j) = 7 + 7j$$

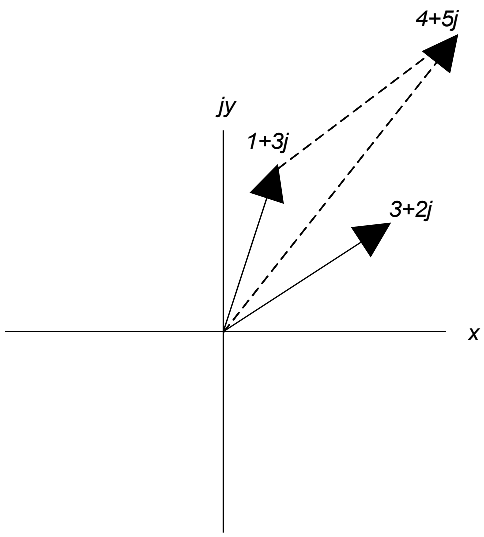

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Figure 2.2a. The addition of vectors

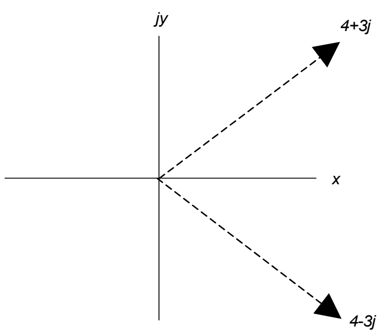

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Figure 2.2b. The addition of vectors

Multiplying a real value with *j* produces an imaginary number and multiplying an imaginary number with *j* results in a real value but in the negative direction. For example, multiply the real value 5 with *j* produces *j5* which is on imaginary axis, subsequent multiplication of *j5* with *j* results in –5 which is on real axis in the negative direction, another multiply of *j* with –5 produces *–j5* which is on negative imaginary axis, subsequent multiply of *–j5* with *j* results back to the real value 5. Figure 2.3 shows how the repeated multiplication with *j* rotates the number in a counterclockwise direction. Similarly, multiplication with *–j* results in a clockwise rotation of 90^o on the complex plane.

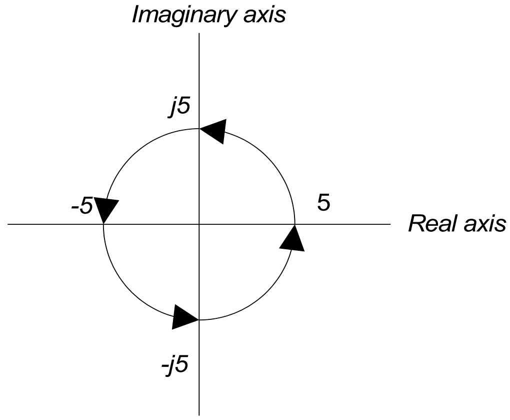

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Figure 2.3. Multiplying with *j* rotates the vector to 90^{o} counter clockwise

A **complex plane** is a Cartesian coordinate system where the distance along the vertical axis is measured in units of $j(j=\sqrt{-1})$, and those along the horizontal axis in the usual units of 1 as shown in Figure 2.4. The y-axis is the imaginary axis and x-axis is the real axis.

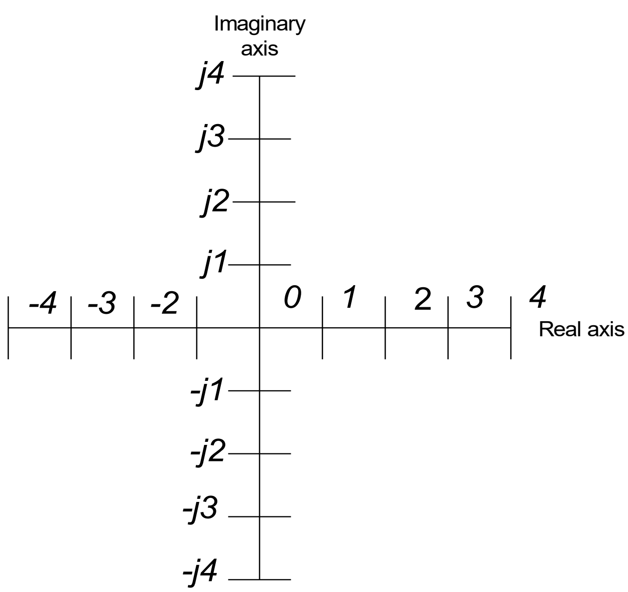

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Figure 2.4. Plot showing imaginary axis and real axis.

Complex numbers in polar coordinates

The position of the complex numbers in Cartesian coordinates determines the distance in x and y directions. Sometimes it is useful to represent the complex number in polar coordinates, where the complex number is represented as a directed line OP, (a vector of length *M*) rotated from the initial line OA through an angle θ. The Figure 2.5 shows a complex number of magnitude vector ***M*** at angle θ.

$$P = M\angle^{\vartheta}$$

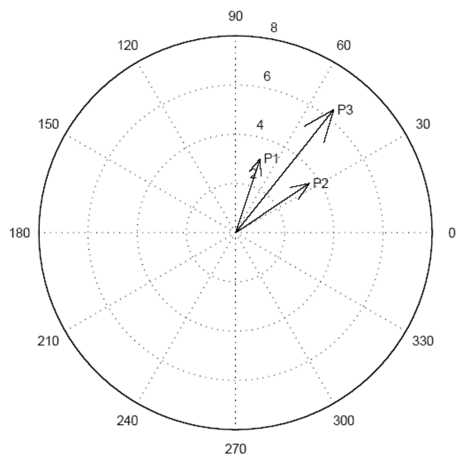

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Figure 2.5. Vector in polar coordinate system

To convert complex numbers in polar coordinates to rectangular form,

$$x = M\cos\vartheta$$

$$y = M\sin\vartheta$$

We can convert the values from the Cartesian coordinates to the Polar coordinates using the trigonometric rule;

$$r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{x}{y}$$

$$x = r\cos(\theta), y = r\sin(\theta)$$

The Polar form of the complex number is,

$$p = r(\cos(x) + j\sin(x))$$

If the point is on a unit radius circle (taking *r* as a unit magnitude) we can write the number as,

$$p = (\cos(x) + j\sin(x))$$

While performing addition and subtraction of complex numbers is easier in rectangular coordinates, there is a definite advantage in doing multiplication and division in Polar coordinates. The rules are simple, for multiplication, multiply the magnitude and add the angles and for division, divide the magnitude and subtract the angles, as shown below,

If $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$

Then,

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)\right]$$

And

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)\right]$$

The multiplication and division can also be performed symbolically,

$$(P\angle^{\alpha})(Q\angle^{\beta}) = PQ\angle^{\alpha+\beta}$$

$$\frac{(P\angle^{\alpha})}{(Q\angle^{\beta})} = \frac{P}{Q}\angle^{\alpha-\beta}$$

$$(P\angle^{\alpha})^{n} = P^{n}\angle^{n\alpha}$$

$$(P\angle^{\alpha})^{1/n} = P^{1/n}\angle^{\alpha/n}$$

The angle θ also referred as the **phase** angle indicates the rotation of the magnitude vector *M* from the real axis as shown in Figure 2.5.

Exponent *e* and the power functions

One of the most useful ways of representing a complex number is the exponent notation. We would be expressing our wave functions using exponents and the complex numbers in exponent form. There will be an extensive use of exponent function in our study of signal analysis, specially, the Fourier Transform, so it is important to grasp the concept of the exponent functions, the following explanation might help bring a picture in mind when you derive a function using exponents.

A product is the result of multiplying two or more quantities; if the quantities are similar then the result is a power function, in which case a variable is raised to a constant power, such as area and volume,

$$Area = l \times l = l^2$$

$$Volume = l \times l \times l = l^3$$

Another form of the power functions is where a constant is raised to a variable. For example, how many ways can you place two numbers in different slots? Obviously you need to how many slots are there, the answer is,

$y = 2^x$. Where x is the number of slots.

For 2 slots the number of combination is $y = 2^2 = 4$ and for 3 slots it is $y = 2^3 = 8$. This is also called a growth function. It is just like money in the bank. You begin with a fixed quantity and it would grow. According to the power factor variable defined by your banker. Commonly known as the interest rate.

There is a simpler method for representing a power function where the power factor of a base number is written as **logarithmic number** also written as **log** for brevity. You will find same logarithmic value for different numbering base. For example, the number 10^6 (1,000,000) could be written as,

$$10^6 = 1000000$$

$$\log_{10}(1000000) = 6$$

And the number 2^6 (64) could be written as,

$$2^6 = 64$$

$$\log_2(6) = 6$$

Except for base 1 any number can be used for logarithmic base. It is customary to omit the base while representing a power number when the base is 10. Followings are the rules of arithmetic operations on logarithmic numbers,

$$\log_a 1 = 0, a^0 = 1$$
$$\log_a a = 1, a^1 = a$$
$$\log_a (M \times N) = \log_a M + \log_a N$$
$$\log_a (\frac{M}{N}) = \log_a M - \log_a N$$
$$\log_a (N^p) = P \log_a N$$

There is a special form of growth function called **exponential growth**, where the growth is proportional to the amount present at the time. This is what you would hear from your stockbroker, "your money would grow exponentially!" What he really meant, his money will grow up exponentially and your money will grow down exponentially. Up or down either way the rate of change is proportional to the amount present at that instance of time.

It is not only money there are several things in nature that follow the trend. Things decay and the rate of decay is proportional to the quantity present at the time, like radioactive materials. Things grow and the arte of growth is proportional to the quantity present at that time, such as build up of electric charge on a capacitor, voltage buildup on an inductor, population growth of living organisms etc.

Mathematically speaking, an exponential function is a power function with the rate of change proportional to the amount present at the time. In other words, with an appropriate proportionality constant the exponent power function's derivative is equal to the power function itself.

$$\frac{dy}{dt} \propto y,$$
$$\frac{dy}{dt} = ky \qquad (2.3)$$

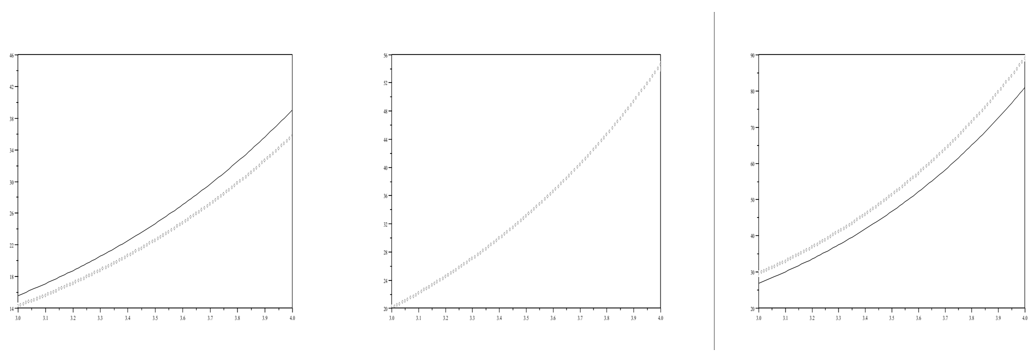
Let's follow the trail and find out what number is that, when raised to a power, gives us a derivative that is same as the number itself as in Equation 2.4:

$$\frac{d(N^t)}{dt} = N^t \qquad (2.4)$$

We can determine the number experimentally by examining few power function graphs. Let's pick three numbers, say, 2.5, 2.718 and 3 as the number base for the power functions and calculate the values of their derivatives at different points. The Table 2.1 shows the tabulation of the power functions of the three numbers and their derivatives and the Figure 2.6 shows how each function differs from its derivative. Comparing the graph of the derivative of the function and the function itself, we find only the number 2.718 has the slope line (dashed line) identical to the power function line (solid line). No other number function has such property, where the derivative is the same as the function itself.

Table 2.1. The tabulation of the power functions and their derivatives.

$t =$	$(2.5)^t$	$\frac{d(2.5)^t}{dt}$	$(2.718)^t$	$\frac{d(2.718)^t}{dt}$	$(3)^t$	$\frac{d(3)^t}{dt}$
3	15.63	14.32	20.08	20.08	27	29.66
3.1	17.12	15.69	22.19	22.19	30.14	33.11
3.2	18.77	17.2	24.52	24.52	33.63	36.95
3.3	20.57	18.85	27.1	27.1	37.54	41.24
3.4	22.54	20.66	29.95	29.95	41.9	46.03
3.5	24.71	22.64	33.1	33.1	46.77	51.38
3.6	27.08	24.81	36.58	36.58	52.2	57.34
3.7	29.67	27.19	40.43	40.43	58.26	64
3.8	32.52	29.8	44.68	44.68	65.02	71.43
3.9	35.64	32.66	49.38	49.38	72.57	79.73
4	39.06	35.79	54.58	54.58	81	88.99



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****** Insert Figure 2.6b here ******
****** Insert Figure 2.6c here ******

Figure 2.6. The graph of power functions and its derivative, a) solid line indicating function $(2.5)^t$ and dotted line indicating function $\frac{d(2.5)^t}{dt}$, b) solid line indicating function $(2.718)^t$ and dotted line indicating function $\frac{d(2.718)^t}{dt}$, solid line indicating function $(3)^t$ and dotted line indicating function $\frac{d(3)^t}{dt}$

The number 2.718 also called **natural number** plays an important role in solving problems in nature involving the rate of change of quantities that follow the natural growth as mentioned earlier. The natural number to be more precise 2.718281828459… is also written as ***e*** and a function that follows exponential growth is defined as e^t. The Equation 2.4 is being rewritten using *e* notation substituting for *N*

$$\frac{d(e^t)}{dt} = e^t$$

A **natural logarithm** (abbreviated with **ln**) is a number when *e* is being used as a base for its logarithmic value. The natural log of *e* is 1, since 2.718 must be raised to the power of 1 to get 2.718 as shown below,

$$e^1 = 2.718$$

$$\ln(e) = \ln(2.718) = 1$$

Any number raised to the power 0 is equal to 1, thus the exponent of 0 is 1 and the natural log of 1 is 0.

$$e^0 = 1$$

$$\ln(1) = 0$$

There was a remarkable discovery made by Euler (1707-1783) that established a relationship between trigonometric functions of sin and cos and the exponent function e^{jt}. This discovery had greatly simplified arithmetic operation and enabled us to describe a complex number in exponent form.

Euler's identity

Euler made an observation that the series expansion of exponent function was equal to the series expansion of sin function and the cos function combined as shown below,

$$\cos\vartheta = 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} - \frac{\vartheta^6}{6!} +$$

$$j\sin\vartheta = j\vartheta - j\frac{\vartheta^3}{3!} + j\frac{\vartheta^5}{5!} - j\frac{\vartheta^7}{7!} +$$

$$\cos\vartheta + j\sin\vartheta = 1 + j\vartheta - \frac{\vartheta^2}{2!} - j\frac{\vartheta^3}{3!} + \frac{\vartheta^4}{4!} + j\frac{\vartheta^5}{5!} + ...$$

$$e^{j\vartheta} = 1 + j\vartheta - \frac{\vartheta^2}{2!} - j\frac{\vartheta^3}{3!} + \frac{\vartheta^4}{4!} + j\frac{\vartheta^5}{5!} + ...$$

We will discuss the exponent function *e* in depth later in the section but for now, we are interested in representing a complex number in the exponent notation. Using Euler's identity, we can denote sin and cos functions in terms of its equivalent exponent form,

$$e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$$

$$e^{-j\vartheta} = \cos\vartheta - j\sin\vartheta$$

$$\cos\vartheta = \frac{e^{j\vartheta} + e^{-j\vartheta}}{2}$$

$$\sin\vartheta = \frac{e^{j\vartheta} - e^{-j\vartheta}}{j2}$$

Going back to our complex value representation, a vector *M* of fixed radial line as shown in the Figure 2.5 can be represented in the exponent form that would allow us to write a sinusoidal function using exponent notations,

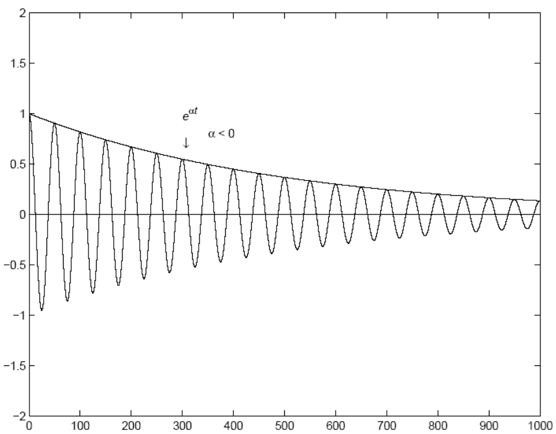
$$W = Me^{j\vartheta} = M\angle^{\vartheta} = (M\cos(\vartheta) + jM\sin(\vartheta))$$

The Complex Frequencies

When a complex number $s = \sigma + j\omega$ appears in an exponential time function e^{st}, s is called the complex frequency.

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

The complex frequencies include an exponential decay factor $\sigma < 0$ (see Figure 2.7a), or exponential rise factor $\sigma > 0$ (see Figure 2.7c), multiplied with a sinusoidal function $(\cos\omega t + j\sin\omega t)$. If (if $\sigma = 0$) then the complex frequencies are pure sinusoid with no decay or rise as shown in Figure 2.7b.



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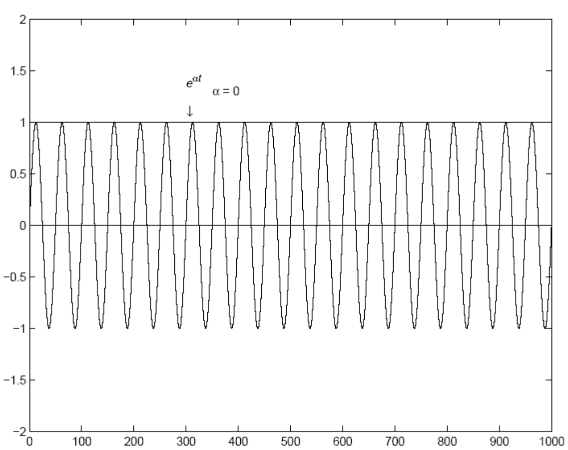

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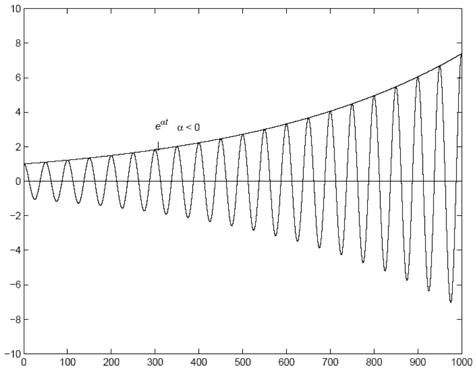


****** Insert Figure 2.7c here ******

Figure 2.7. The varying combinations of complex sinusoids, a) Exponential rise, b) No Exponent, c) Exponential decay

Phase Angle

If a sin function starts with an angle other than 0 then it is considered having a phase angle θ that represents a time delay in the sinusoid. If ω is the frequency then ωt is the angle that is formed at the particular instance of time *t*. If the frequency function $\sin(\omega t)$ has a phase angle θ then the sin function must be represented as $\sin(\omega t + \theta)$. It can also be shown as a function of the complex point $W = Me^{j\theta}$ as if the line *M* rotating with an angular velocity ω as shown in Figure 2.8c.

$$W(t) = Me^{j(\omega t + \theta)} = Me^{j(\omega t)} \times Me^{j(\theta)}$$

Thus, a sinusoidal function is a complex constant $Me^{j\vartheta}$ (amplitude M and the phase angle θ) also known as **phasor**, multiplied by a function of time $Me^{j(\omega t)}$ indicating rotation with an angular velocity ω. Electrical circuit components such as capacitors and inductors do not alter the frequency of the input signals; they only affect the amplitude of the input wave or change the phase angle of the input frequency. The quantity $Me^{j(\omega t)}$ is usually omitted during network response calculations, since frequency remain same.

In terms of frequency $f = \omega / 2\pi, \omega = 2\pi f$

$$W(t) = Me^{j(2\pi f + \vartheta)}$$

The projection of the line *M* on the real axis is

$$W_{(real)} = f(t) = (M\cos(\omega t + \vartheta))$$

And on the imaginary axis

$$W_{(imag)} = f(t) = (M\sin(\omega t + \vartheta))$$

If an instantaneous voltage is described by a sinusoidal function of time such as

$$v(t) = V\cos(\omega t + \vartheta)$$

Then $v(t)$ may be interpreted as the real part of a complex function or

$$v(t) = \mathrm{Re}(Ve^{j(\omega t + \vartheta)}) = \mathrm{Re}(Ve^{j\omega t} \times Vej^{j\vartheta})$$

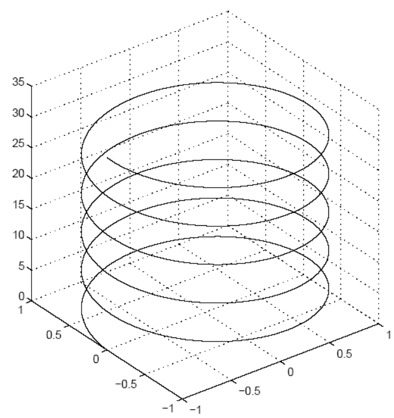
The Figure 2.8a is a 3 dimensional space representation of a sinusoidal function showing x-axis as the real value angular frequency ω, the y-axis as the imaginary value of $j\omega$ and time *t* in the z direction. If we look though the graph from the z direction then the sinusoidal function is represented as a point on a circle. A sin wave is a function $e^{-j\omega t}$ moving in a counter-clockwise direction and a cos wave is a function $e^{j\omega t}$ moving in a clockwise direction, both having the same amplitude and the phase angle. Thus the periodic function is a function of complex number in continuous time,

$$y(t) = e^{j(\omega t + \vartheta)}$$

$$y(t) = e^{-j(\omega t + \vartheta)}$$

Conjugate function

The negative exponent e^{-jx} is also called conjugate of the positive exponent e^{jx}. The two functions are mirror image of each other on a complex plane as shown in fig 2.8.



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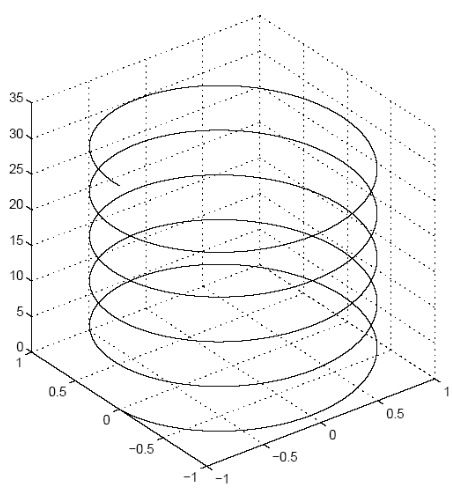

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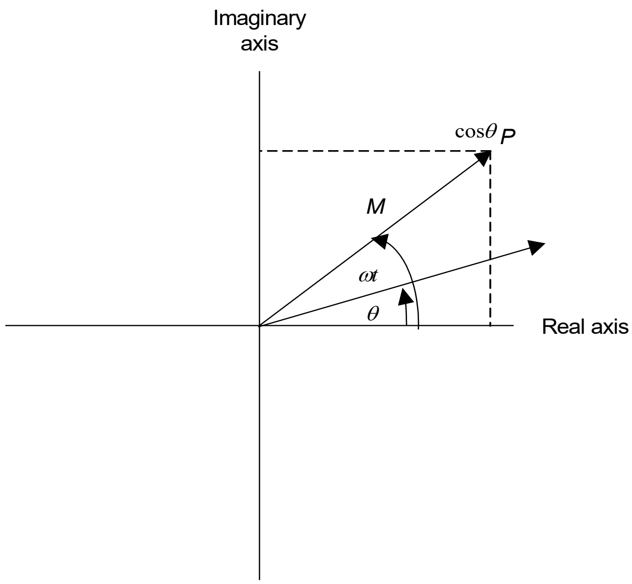

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Figure 2.8.a. The clockwise direction of the function e^{-jx}, b) The counter-clockwise direction of the function e^{-jx}.

The advantage of phasor notation will be obvious in the next section as we try to analyze electrical networks.

The phasor method

The phasor method is a method of solving electrical network problems where the current and voltage excitations applied to the networks are all sinusoidal. Before we solve these problems, a brief description of behavior of the electrical components is given as a refresher.

The electrical networks

Most of the digital signal processing was derived from the analog world and in some situation we would like to replace the functionality of the electrical circuits using the software algorithms, so a thorough understanding is necessary. We will study how resistors, capacitors and inductors behave as filters of frequencies and integrate and differentiate the incoming signals of the analog world. During this study, we will assume an ideal behavior of the electrical components, where there is no loss of energy due to heat etc.

Resistors

The voltage on a resistor is directly proportional to the current applied

$$V_r = I_r R$$

Capacitors

An electric charge builds up on the dielectric plates of a capacitor in response to an applied voltage. The charge is proportional to the voltage applied and the capacitance *C* is the proportionality constant.

$$q = Cv$$

The rate of the charge buildup (known as current) is proportional to the rate of change of voltage.

$$\frac{dq}{dt} = i = C\frac{dv}{dt} \qquad (2.5)$$

For a sinusoidal excitation voltage

$$v = e^{j\omega t}, \frac{dv}{dt} = j\omega e^{jwt} \qquad (2.6)$$

Substituting the value of Equation 2.6 into Equation 2.5 we get,

$$v = \frac{1}{j\omega C} i$$

Inductors

A rapidly changing current induces a voltage across a coil made of conductive material such as copper. The voltage is proportional to the rate of change of current and inductance *L* is the proportionally constant.

$$v = L\frac{di}{dt} \qquad (2.7)$$

For a sinusoidal current

$$i = e^{j\omega t}, \frac{di}{dt} = j\omega e^{jwt} \qquad (2.8)$$

Substituting the value of Equation 2.8 into Equation 2.7 we get,

$$v = j\omega L i$$

The Table 2.2 summarizes the relationship between the input excitation to the transformed output. The **impedance Z** is the direct proportionality constant and the **admittance Y** is the inverse proportionality constant.

Table 2.2.The table showing voltage and current relationship in terms of impedance and admittance of the three circuit elements.

Input excitation $v = V_m \cos(\omega t + \vartheta)$ and $i = I_m \cos(\omega t + \alpha)$

	Impedance	Admittance
Resistor	$V_r = I_r R$	$I_r = V_r G$
	$R = \frac{1}{G}$	$G = \frac{1}{R}$

Capacitor	$V_c = \frac{1}{j\omega C} I_c$	$I_c = j\omega C V_c$
	$jX = \frac{1}{j\omega C}$	$jB = j\omega C$
Inductor	$V_l = j\omega L I_l$	$V_l = \frac{1}{j\omega L} I_l$
	$jX = \frac{1}{j\omega L}$	$jB = j\omega L$
	$V_l = ZI = (R + jX)I_l$	$V_l = YI = (R + jB)I_l$

Example 2.1

Describe the voltage $v(t) = V_{in}\cos(\omega t + \vartheta)$ in terms of phasor.

$$v(t) = V_{in}\cos(\omega t + \vartheta)$$
$$v(t) = \mathrm{Re}\{V_{in} e^{j(\omega t + \vartheta)}\} = \mathrm{Re}\{(V_{in} e^{j(\vartheta)})(V_{in} e^{j(\omega t)})\}$$
$$v(t) = V_{in} e^{j(\vartheta)} = V_{in}\angle\vartheta$$

Example 2.2

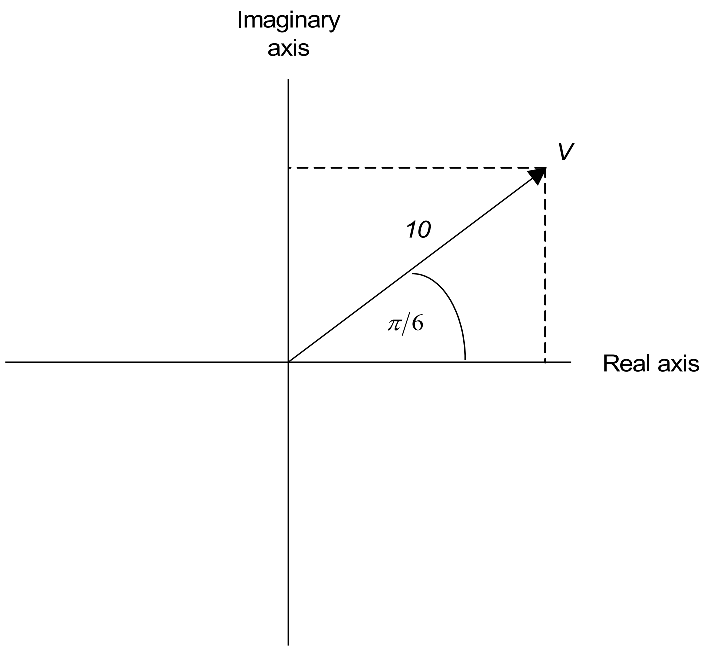
Represent the voltage $v(t) = 10\sin(\omega t + \pi/6)$ in phasor notation.

By definition the phasors are represented as the cosine part (the real part) of the complex function in time. Since $\cos(\vartheta - \pi/2) = \sin(\vartheta)$, we need to convert our sin function into cos equivalent.

$$v(t) = 10\cos(\omega t + \pi/6 - \pi/2) = 10\cos(\omega t - \pi/3)$$

The Figure 2.9 is the graphical representation of the voltage phasors

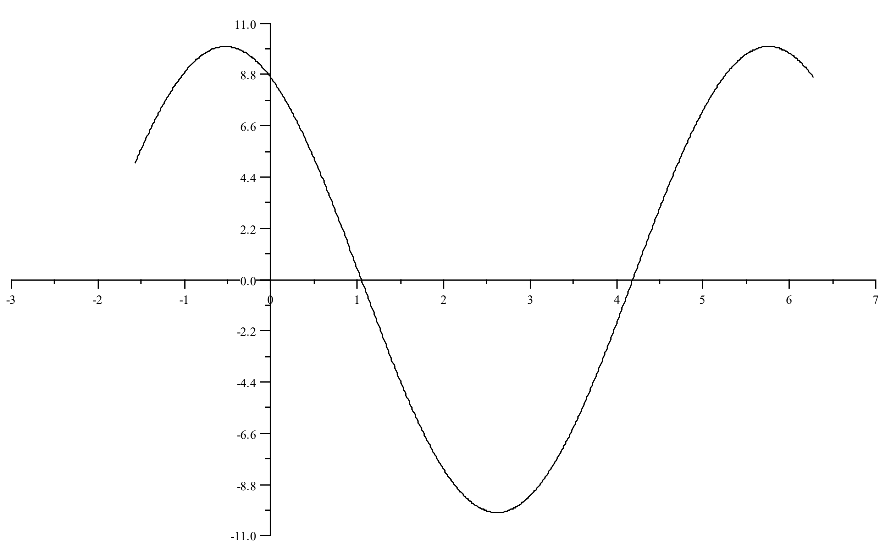
$$V = 10e^{-j(\pi/3)}$$



****** Insert Figure 2.9 here ******

Figure 2.9. The voltage phasor of Example 2.2

Example 2.3
If the current across the capacitor in a network is a function of time as shown in Figure 2.10. Write the equation of the current.



****** Insert Figure 2.10 here ******
Figure 2.10. The voltage function for the Example 2.3

A complete cycle is achieved in 1 second, thus,

$$f = 1cycle/\sec =$$

$$2\pi radians = 1cycle$$

$$\omega = 2\pi radians/\sec$$

The current reaches its maximum value of 10 at 30 degree before the time t=0, thus the phase angle

$$\vartheta = \pi/6 = 30\deg$$

The instantaneous current is

$$i(t) = 10\cos(2\pi t + \pi/6)$$

$$i(t) = \mathrm{Re}\{10e^{j(\omega t+\vartheta)}\} = \mathrm{Re}\{10e^{j(2\pi t)}\} = \mathrm{Re}\{10e^{j\vartheta}\}$$

The phasor I

$$I = \mathrm{Re}\{10e^{j\vartheta}\}$$

$$I = 10e^{j(\frac{\pi}{6})} Amp$$

Example 2.4
The sinusoidal 60 Hz AC input voltage to the circuit of Figure 2.11 is 110 V at its peak when *t=0.* Describe the instantaneous voltage in complex notation form.

$$V_{\max} = 110$$

$$\omega = 2\pi 60$$

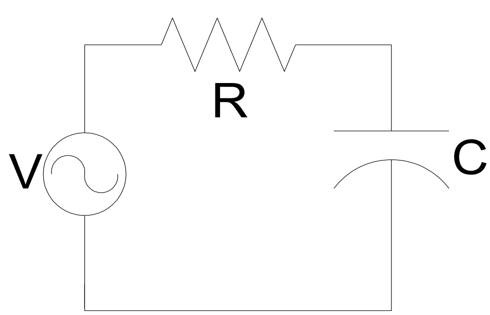
$$\vartheta = 0$$

$$v(t) = V_{\max}\cos(wt + \vartheta) = 110\cos(2\pi 60t)$$

$$v(t) = \mathrm{Re}\{Ve^{j(\omega t+\vartheta)}\} = \mathrm{Re}\{(Ve^{j(\vartheta)})(e^{j(\omega t)})\}$$

$$v(t) = 110e^{j(2\pi 60t)}$$

$$v(t) = 110\angle^{0}$$



****** Insert Figure 2.11 here ******

Figure 2.11. The Voltage across the RC circuit for the Example 2.4

Example 2.5

Find the voltage V_{out} across the capacitor C of Figure 2.11 as a function of time for an input excitation of $v(t) = V_{in}\cos(\omega t + \vartheta)$

Note: Determine the current through the loop and then calculate the voltage drop across the capacitor.

The sum of the voltage drop across the resistor and the capacitor is equal to the input voltage applied.

$$V_{in} = V_R + V_C$$

$$V_{in}\cos(\omega t + \vartheta) = IR + I\frac{1}{j\omega C}$$

The current I through the loop

$$I = \frac{1}{R + \frac{1}{j\omega C}} V_{in}\cos(\omega t + \vartheta)$$

The output voltage at the capacitor

$$V_{out} = \frac{1}{j\omega C} \times \frac{1}{R + \frac{1}{j\omega C}} V_{in}\cos(\omega t + \vartheta)$$

$$V_{out} = \frac{1}{j\omega C + 1} V_{in}\cos(\omega t + \vartheta)$$

The magnitude of the output voltage

$$V_{out} = \frac{1}{\sqrt{1 + (\omega RC)^2}} e^{j\vartheta} = \frac{1}{\sqrt{1 + (\omega RC)^2}} \angle \tan^{-1}(\omega RC) \qquad (2.9)$$

The phase angle

$$\vartheta = \tan^{-1}(\omega RC) \qquad (2.10)$$

Example 2.6

In the circuit of Figure 2.11, input voltage is a 60 Hz AC, C=2 and R=2. Calculate the phase angle and the voltage on the capacitor C.

Substituting the values into equation 2.9

$\omega = 2\pi f = 120\pi$

$V_c = \frac{1}{\sqrt{1+(2\times 2\times 120\pi)^2}}\cos(2\pi 60t)$

The phase angle ϑ

$\vartheta = \tan^{-1}\omega RC = \tan^{-1}(4\times 120\pi)$

Example 2.7

Find out the frequency at which the output voltage reaches 70 % of the input voltage in the circuit of Figure 2.11.

Substituting the values into Equation 2.8:

$$\frac{V_{out}}{V_{in}} = 0.7 = \frac{1}{\sqrt{1+(j\omega RC)^2}}$$

$$0.5(1+\omega RC) = 1$$

$$\omega RC = 1$$

The cutoff frequency is

$$\omega = \frac{1}{RC} \qquad (2.11)$$

Note: A **cutoff frequency** is the input frequency at which the output is 0.707 of the input.

Example 2.8

Determine the phase angle at the cutoff frequency in the circuit of Figure 2.11.

Substituting the values into Equation 2.9:

$\vartheta = \tan^{-1}(-1) = -45^0$

Note: at cutoff frequency the phase angle is 45^0

.

Low Pass Filter

The Equation 2.9 reveals a peculiar property of the circuit in Figure 2.11 that at higher frequencies the output voltage gain is considerably lower then at lower frequencies due to the multiplying factor $\frac{1}{\sqrt{1+(j\omega RC)^2}}\times V_{in}$. The circuit could be used to discriminate lower frequencies, as if lower frequencies are passing without much degradation. The circuit of Figure 2.1 is a simple low pass filter.

Differentiation

The circuit of Figure 2.8 is also a differentiator. The current across a capacitor is proportional to the rate of change of voltage. The same current is being fed to the resistor. Thus the output is approximately proportional to the derivative of the input voltage.

$$v = C\frac{dv}{dt}$$

We will discuss the differentiation and filtering more in the upcoming chapter of analog filter design.

Example 2.9

A 110 Volt 60 Hz AC input is applied to the circuit of Figure 2.11, with C=2 Farads and R=2 Ohms. Determine the ratio of the output voltage across the capacitor to the input voltage.

Substituting the values into Equation 2.8:

$$V_{in} = 110\cos(2\pi 60t + 0)$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{1+(4\times 240\pi)^2}}$$

Example 2.10

If the frequency is reduced to 1 Hz in the previous example of 2.10, determine the improvement in the output voltage across the capacitor to the input voltage.

Substituting the values into Equation 2.8:

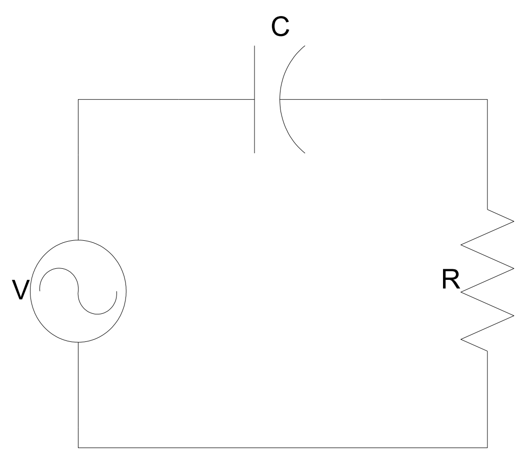
$$V_{in} = 110\cos(2\pi t + 0)$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{1+(4\times 4\pi)^2}}$$

Nearly a 60 times improvement in the output at 1 Hz compare to the 60 Hz input.

Example 2.11

Determine the voltage and the phase angle across the resistor R in Figure 2.12 for an input voltage of $V_{in}\cos(\omega t + \vartheta)$



****** Insert Figure 2.12 here ******

Figure 2.11. The Voltage across the RC circuit for the Example 2.11

Similar to the Example 2.5 we need to determine the current through the loop and through that we can calculate the voltage across the resistor. The sum of the voltage drop across the resistor and the capacitor is equal to the input voltage applied.

$$V_{in}\cos(\omega t+\vartheta)=IR+I\frac{1}{j\omega C}$$

$$I=\frac{1}{R+\frac{1}{j\omega C}}V_{in}\cos(\omega t+\vartheta)$$

$$V_{out}=\frac{R}{R+\frac{1}{j\omega C}}V_{in}\cos(\omega t+\vartheta)=\frac{1}{1-j\frac{1}{\omega RC}}V_{in}\cos(\omega t+\vartheta)$$

The magnitude of the voltage across the resistor is

$$V_{out}=\frac{1}{\sqrt{1-\frac{1}{(\omega RC)^2}}}V_{in}\cos(\omega t+\vartheta)=\frac{1}{\sqrt{1-\frac{1}{(\omega RC)^2}}}V_{in}e^{j\vartheta} \tag{2.12}$$

The phase angle

$$\vartheta=\tan^{-1}(-\frac{1}{\omega RC}) \tag{2.13}$$

High Pass Filter
Compare the Equation 2.13 with the Equation 2.9. The affect is now reverse; at higher frequencies the output voltage gain is considerably higher then at lower frequencies (the multiplying factor $\frac{1}{\sqrt{1-\frac{1}{(\omega RC)^2}}}\times V_{in}$). The circuit could be used to discriminate higher frequencies, as they pass through with less degradation. The circuit of Figure 2.12 is a simple high pass filter.

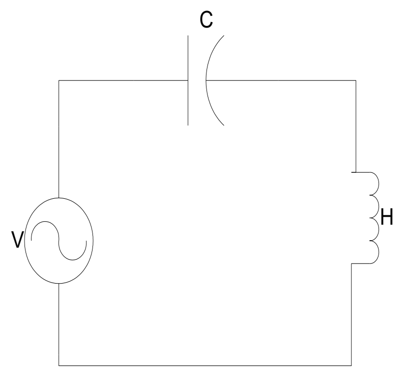
Integration
The circuit of Figure 2.7 is also an integrator since the voltage across a capacitor gradually builds up in time. The output is approximately proportional to the integral of the input.

$v = \frac{1}{C}\int idt$

We will discuss integration more in the upcoming chapter of analog filter design.

Example 2.12

In the circuit of Figure 2.13 the inductor L is 2 mH and the capacitor C is 4 uF. At what frequency the total impedance will be 0.



****** Insert Figure 2.13 here ******

Figure 2.11. The LC circuit for the Example 2.12

The two imaginary components in the circuit are the capacitive reactance $\frac{1}{j\omega RC}$ and the inductive reactance $j\omega L$. There is no resistor so real component is 0. The impedance is

$$Z = \sqrt{(0) + (\omega L - \frac{1}{\omega C})}$$

The impendence will be 0 when $\omega L = \frac{1}{\omega C}$.

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2\times 10^{-3} \times 4 \times 10^{-6}}}$$

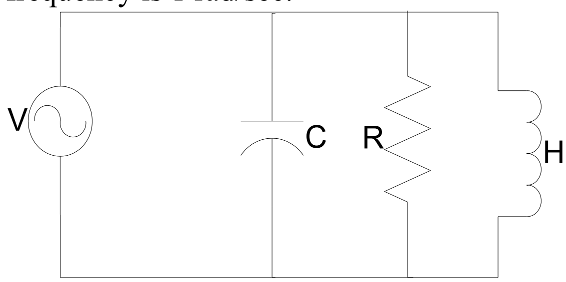
Resonance

Having a 0 impedance indicates an infinite gain since the output voltage is supposed to have been divided by the impedance, which is now 0. But of course, in reality we don’t have ideal components, there is a certain amount of resistance in every inductor and in every capacitor. But the point is, when a capacitor is connected with an inductor in series there is always a frequency at which the both reactance just cancel each other and provide maximum gain at the output. The frequency at which the capacitive reactance is equal to the inductive reactance is

called the **resonance** frequency. At resonance the only impedance is due to the resistive components.
The resonance frequency is computed as

$$\omega = \frac{1}{\sqrt{LC}}$$

Example 2.13
Use the phasor method to determine the current in the elements of the circuit of Figure 2.14. $C = 2F, R = 2\Omega, L = 2H$. The input voltage is $110\angle^{0} V$. and the frequency is 1 rad/sec.



****** Insert Figure 2.14 here ******
Figure 2.11. The RLC circuit for the Example 2.13

$$\omega = 1$$

$$I_R = Y_R V = \frac{1}{2} \times 110\angle^{0}$$

$$I_c = Y_C V = j\omega C V = j2 \times 110\angle^{0} = 220\angle^{90}$$

$$I_c = Y_C V = \frac{1}{j\omega L} V = -j \times 55\angle^{0} = 55\angle^{-90}$$

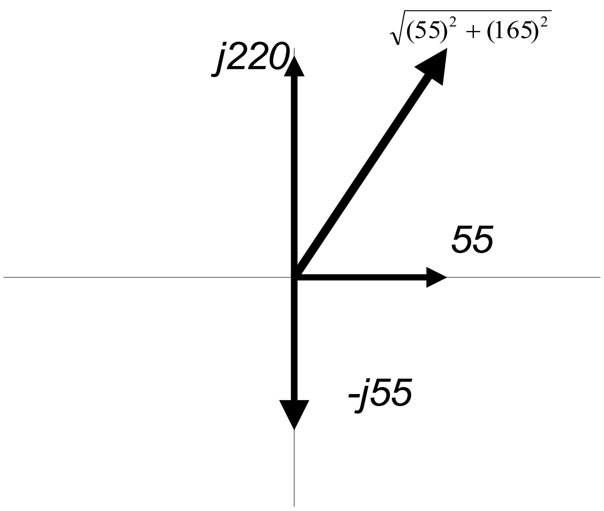
Notice the multiplication of *J* is the same as a phase angle 90^0 and the multiplication of -*J* is the same as a phase angle -90^0.

Example 2.14
Use the phasor method to determine the total current through the elements of the circuit of Figure 2.11. $C = 2F, R = 2\Omega, L = 2H$. The input voltage is $110\angle^{0} V$. and the frequency is 1 rad/sec.

The total current is the sum of the individual currents. The Figure 2.15 shows the graphical addition the current phasors.

$I_T = I_R + I_C + I_L$

$$I_T = I_R + I_R + I_R$$
$$I_T = 55\angle^{0} + 220\angle^{90} + 55\angle^{-90}$$
$$I_T = 55 + j0 + 0 + j220 + 0 - j55$$
$$I_T = 55 + j165 = \sqrt{(55)^2 + (165)^2}\angle^{\tan^{-1}\frac{165}{55}}$$



****** Insert Figure 2.15 here ******
Figure 2.15. The graphical method of computing current phasors

Summary

A complex number is a vector on a coordinate system representing amplitude and phase angle of a sinusoidal function. The two parts of a complex number are the real part (the x distance from the origin) and the imaginary part (the y-distance from the origin.) The imaginary part carries the operator $\sqrt{-1}$ that performs the necessary algebraic manipulations of the imaginary part. The three different ways of representing a complex number namely the trigonometric form *(x+jy),* the polar form $P = M\angle^{\vartheta}$ and the exponent form $Me^{j\vartheta}$ were discussed. The phasor method was developed (based on the complex number notation) to solve the electrical network problem where the input excitation is a sinusoidal form. The chapter gives you a brief overview of the problems that our digital signal processing is supposed to solve.