Chapter 1 Fourier Analysis

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Introduction

We begin our study of DSP with an introduction to Fourier analysis. Although a somewhat complex beginning but it is necessary to create a mind set which is different from our every day perception of events. By events we mean physical signals that are measurable, such as temperature, pressure, voice and sound etc. These signals, though arbitrary in nature, must be defined as mathematical functions if further processing is required. The processing of the signals could mean anything, from removing the unwanted noise from the music to designing a control system for maintaining a constant action. The central idea of the Fourier analysis is to look through these events as if they are functions made up of superimposed sinusoidal frequencies. How to identify these component frequencies is the topic of discussion in this chapter.

The 17^{th} century mathematician Jean Baptiste Joseph **Fourier** (1768-1830) discovered that a continuous time **periodic function** (the one that repeats a pattern periodically after certain interval) could be approximated as a series of simple sin and cosine functions and in his honor, such an approximation of the function is called the **Fourier series** of the function. Although it is hard to find functions in nature that are true periodic for all time, we could still define a proper domain of interest and approximate a function using its Fourier series. But when we deal with an arbitrary signal we don’t know its period, later on in chapter 3 we will see how we can remove this periodic dependency, we obtain what we call the relative magnitudes of the component frequencies, and that would be the **Fourier Transform** of the signal. In essence, the concept of Fourier analysis provides us a different domain, the domain of frequency to analyze a signal.

Frequency analysis is important in many situations as systems behave differently at different frequencies. Imagine a child on a swing; the swing is a system and it requires an external periodic force for its operation. Only when the frequency of the external force matches the natural frequency of the swing, the swing will keep going, otherwise, it will stop. If we knew the Fourier series of the external force beforehand, we could predict weather and how long the swing will move. When the component frequencies of a function are identified, the overall processing of the system could be evaluated as the sum of the responses of the individual frequencies, as if, each was a separate input applied to the system. For some class of functions further processing is only possible once the component frequencies are identified, this would be evident from the example presented at the end of the chapter.

The Fourier series and Fourier Transform are not the only aspect of DSP and it is not necessary to transform all functions into its frequency components, but the concept is important and you should be able to deal with it when it is necessary. In this chapter we will be developing techniques for analyzing and formulating a waveform as described by Fourier.

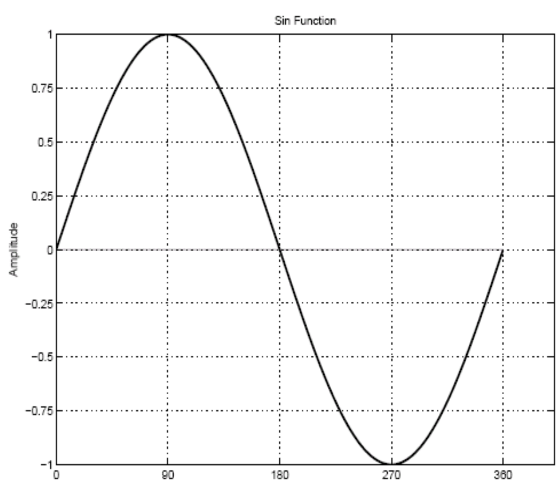
Periodic waves

By definition a function is periodic if its basic pattern is repeated with a predefine time interval or the period. The period is simply the time between each crust of the wave and is fixed for a true periodic wave. Fourier showed us that a seemingly arbitrary waveform is basically composed of several simple sinusoidal waves. The unique thing about his discovery was that not only he showed us how to compute the exact number of these simple waves but also devised the method of extracting the height of each individual wave.

(Please note that the word wave is being used in the context of sinusoidal waves and not the traveling waves at a distance)

A simple periodic function

Physical events may be classified as periodic or non-periodic. Any repeated pattern could be considered as a periodic function, but a simple periodic function is a cyclic pattern that has fixed amplitude and a fixed period as shown in Figure 1.1. Think a bout a bicycle having a painted spot on its wheel. The graph of the spot as the wheel moves may be described as shown in Figure 1.3. The path traversed by the spot on the wheel is an example of a simple wave. When the wheel makes a complete revolution the spot achieves a peak and a valley and comes back to its original height. As time progresses the same pattern is repeated. It start at 0^o degree achieves a full height at 90^o goes back to 0 height at 180^o, at 270^o it reaches its minimum height and then goes back to 0 at 360^o.



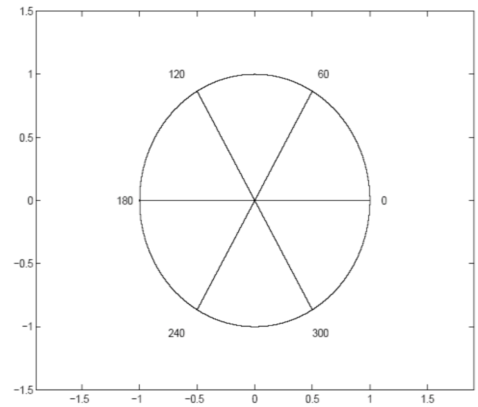
******* Insert Figure 1.1 here******

Figure 1.1 A simple periodic function in time.

The two important aspects are the maximum height and the period of revolution. For a simple wave the amplitude and the period are fixed. The period tells us the time it takes to make a complete revolution and the amplitude tells us the maximum height the wave can achieve. The inverse of period is frequency indicating how many revolutions are completed in one second. This is a typical cyclic function, x direction is the angle of rotation the independent variable and y direction is the amplitude or the dependent variable. Algebraically speaking, a periodic function is a function of angular displacement in time and the amplitude of the wave. The amplitude attains one maximum high and one maximum low during one complete cycle.

Mathematical description of a wave function

Before we begin to describe an arbitrary wave let's discuss how a simple sinusoidal wave is being formed as a function of time. A function is a relationship of two or more quantities described algebraically in a mathematical form. In our previous example of the motion of the spot on the bicycle wheel there is a direct relationship between the height of the spot and the rotation of the wheel. Angles are measured in degrees, if a circle is sliced into 360 equal parts, each is 1 degree of rotation and angles are measured in radians if the arc of rotation is equal to the length of its radius itself. There are exactly 2π radians in one complete cycle, as shown in the Figure 1.2. One radian is slightly less than 60 degrees.



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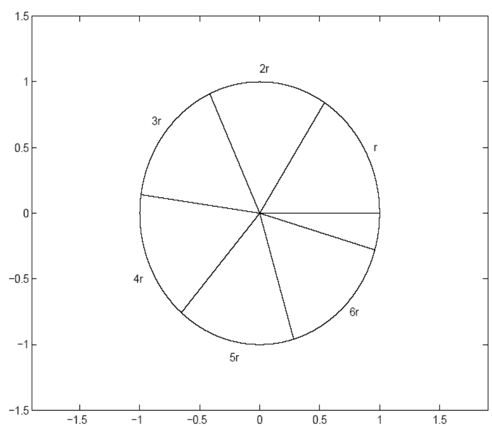

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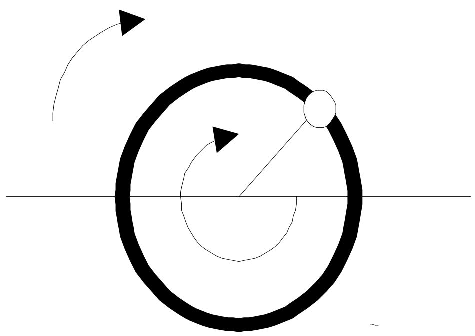
Figure 1.2a. A circle is divided into a) 2pi radians b)360 degrees.

Going back to the bicycle example, the motion of the spot can be described in two ways, a) The change in the height of the spot with respect to the angular movement of the wheel and b) change in the position of the spot with respect to time. There are three variables in our function, the height *h*, the angular distance θ and the time *t*. First, let's develop the functional relationship between the height and the angular displacement θ.



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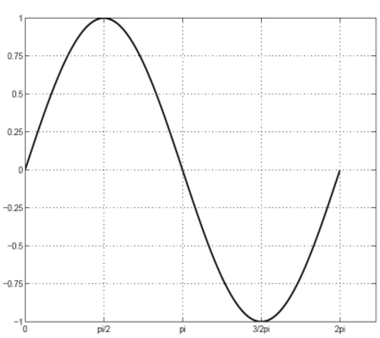

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Figure 1.3.a) Wheel with a spot, b) Motion of the spot as the wheel moves in time

Sin and Cos functions

Looking at the rotation of the bicycle wheel, starting from 0 angle the spot is at 0 feet height and the change in the pattern of the height *h* with respect to the angle θ is shown in the Figure 1.4.

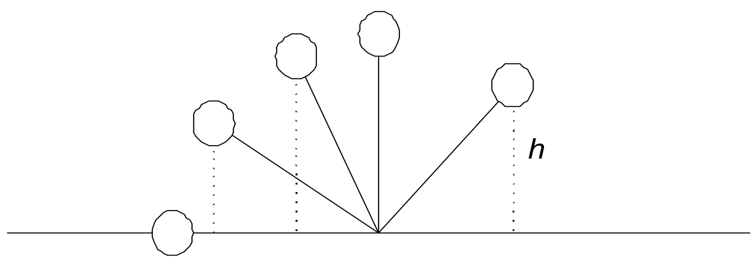

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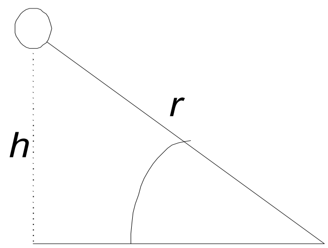
Figure 1.4. Height *h* as a function of the angle θ

The spot gradually increases in height as we increase the angle θ, until it reaches its full height of 1 foot that is the radius of the wheel, exactly at $\pi/2$ radians, further increase in angle brings down the height until it reaches 0 at π radians, completing the half cycle. Further increase in the angle results in the increase in height but in the opposite direction, reaching maximum negative height at $3\pi/2$ radians. From $3\pi/2$ radians to 2π radians the height comes back to zero where we started. The cycle is repeated forever as the angle is increased further. Trigonometrically, the relationship of the height *h* and the angular displacement θ can be described as a sin function as shown in Figure 1.5,

$$h = f(\theta)$$

$$h = r \times \sin(\theta)$$



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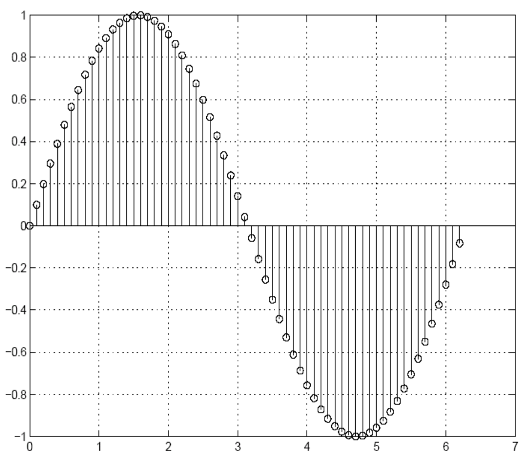

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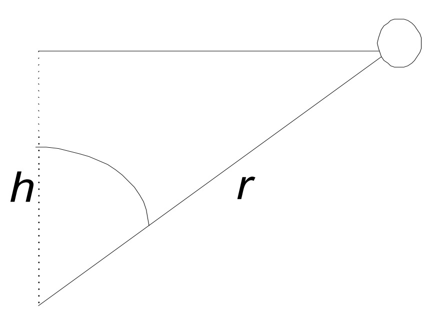
Figure 1.5.a Height of the spot as a sin function of the angle θ , b) Projection on the x axis

If we start the cycle when the height is at the top then the trigonometric relationship is defined in terms of cos function as shown in the Figure 1.6.



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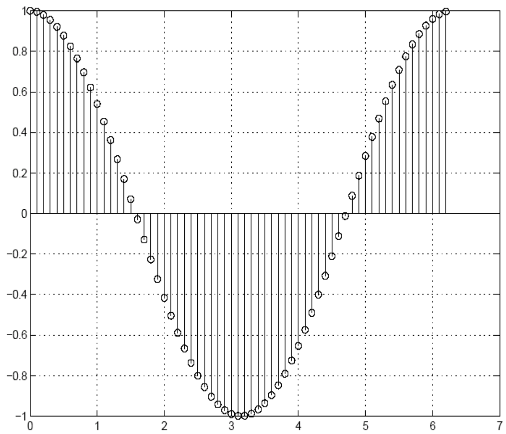

***** **Insert Figure 1.6b here******

Figure 1.6. a) Graph of cos function, showing height vs angle. B) Height *h* as a cos function of the angle θ

$$h = r \times \cos(\theta)$$

Sin and cos functions are identical except the pattern of the cos function is delayed by 90^o. The terminology for describing the angular difference between the two functions is called **phase delay**, thus the phase delay between sin and cos is 90^o.

Angular velocity and the frequency of a periodic function

Looking at the motion of the bicycle wheel, the angular displacement in itself is a function in time. It could be described as frequency *f* with units of cycles/second or angular velocity ω as radians/sec. The **amplitude** of a wave is its maximum height, which is essentially the radius *r* of the circle formed by the circular motion.
If the processing is performed with the help of an analog circuitry then the signal in time is considered **continuous time signal**, since signal flow is continuous in analog circuitry, but if a digital computer is being used then it is a **discrete time signal**, as there is a certain amount of time between successive sampling and processing of the signal.

Frequency

$$f = cycle/\sec$$

If the period of one complete cycle is *T* seconds then,

$f = \frac{1}{T}$ Cycles/sec.

Angular velocity

When velocity is measured as change of angle per unit time then,

$$\omega = \frac{\theta}{t}, radians/\sec$$
$$\theta = \omega t, radians$$

A complete cycle is 2π radians, thus the angular velocity in radians is given as,

$$\omega = 2\pi f$$
$$\omega = \frac{2\pi}{T}$$

Substituting the angle θ for the angular displacement in time $\theta = \omega t, radians$ we can describe the mathematical formulation of a **simple periodic wave** in terms of a sin function as well as a cosine function,

$$h = r \times \sin(\omega t)$$
$$h = r \times \cos(\omega t)$$

If the angular displacement is described in frequency then we must convert it into radians for further calculations,

$$h = r \times \sin(2\pi f t)$$
$$h = r \times \cos(2\pi f t)$$

If we assume the bicycle is moving with a velocity of 1 ft/sec and the spokes of the wheel are one foot long then a complete revolution of 2π feet is achieved in 2π seconds, or the angular velocity ω of 1 radian per second or the frequency $1/2\pi$ cycle /sec.

$$r = 1$$
$$\omega = \frac{\theta}{t} = 1 radian/\sec$$

$$f = \frac{1}{2\pi}\omega = 0.159 cycles/\sec$$

Properties of simple periodic waves

The two distinctive properties of simple periodic functions are the amplitude *h* and the angular velocity ω. A specific simple wave is different from all other simple waves if its amplitude and the angular velocity ω are different from all others. It is customary to differentiate waves by using subscript variable *r* and ω.

$$f_1(t) = r_1 \times \sin(\omega_1 t)$$
$$f_2(t) = r_2 \times \sin(\omega_2 t)$$
$$f_3(t) = r_3 \times \sin(\omega_3 t)$$
$$f_4(t) = r_4 \times \sin(\omega_5 t)$$
$$\vdots$$
$$f_n(t) = r_n \times \sin(\omega_n t)$$

Arbitrary Waveform

You may have experienced that plotting data from a measurement do not usually follow a definite path of a curve, but it is not too difficult to figure out after a few observations if there is a repeated pattern in the data, and when one is present, this could be described as an arbitrary waveform. Even if there is no repeated pattern we could still call it an arbitrary waveform, as we can argue that we didn't have enough time to monitor the event to see its repeated pattern. In any case, treat the data in hand as the data for one complete cycle. What we are interested in is to describe the waveform mathematically in an algebraic form using trigonometric functions, in this description the sin and cosine functions are the building blocks of the waveform and this is what Fourier analysis is all about, giving an event a mathematical description as function of frequency.

Converting a function into its Fourier series is a reversible process. If a function is decomposed into its component frequencies then adding these frequencies would bring back the original function. The former is Fourier analysis and the latter is Fourier synthesis. In either case, we need to determine the amplitude, the period and the total number of trigonometric functions that describe the waveform. It is easier to identify the period and the total number of the trig functions (that you will see next when we discuss Fourier synthesis), but the difficult part is identifying the amplitude of the sin and cosine waves and that will be deferred till the subsequent section of the Fourier analysis.

Fourier synthesis

The staring point for synthesizing a given arbitrary waveform is to determine the fundamental frequency of the sin and cosine functions that make up the waveform.

A waveform that starts with 0 heights may only have the sin components and not the cosine components; since the presence of cosine functions would force the waveform to start with a value other then 0 and that would defeat the purpose of synthesizing the desired waveform. Also notice; the period of one complete cycle of the waveform should be same as the period of the sin function. Otherwise we would end up with the situation where ends don't meet. Thus, the first sin function has the period equal to the period of the arbitrary waveform.

With these assumptions we denote the fundamental sin function as $a\sin(\omega t)$, where a is the amplitude or the **coefficient** of the sin wave and the ω is the angular frequency equal to the frequency of the waveform.

A single sin function is not sufficient, so we add some more. Obviously the next function should have a different period. But notice that the new sin wave should also meet the same criteria that it must start at 0 and end at 0. This can only be achieved if the period of the next sin function is an **integer multiple** of the first one. Otherwise we will end up with the same predicament that the ends don't meet and that would formulate a different function. (We will explore how to determine the total number of subsequent sin functions in the next section)

We denote the fundamental sin wave and all subsequent sin waves as;

$$
\begin{aligned}
&a_1 \times \sin(\omega t) + \\
&a_2 \times \sin(2\omega t) + \\
&a_3 \times \sin(3\omega t) + \\
&\vdots \\
&a_n \times \sin(\omega_n t) \\
&= \sum_{n=1}^{\infty} a_n \times \sin(n\omega t)
\end{aligned}
$$

For example, an arbitrary waveform function $f(t)$ as shown in the Figure 1.7 may be synthesized by adding the following three sin functions

$$f(t) = 1.5 * \sin(t) + 0.5 * \sin(2t) + 0.3 * \sin(3t)$$

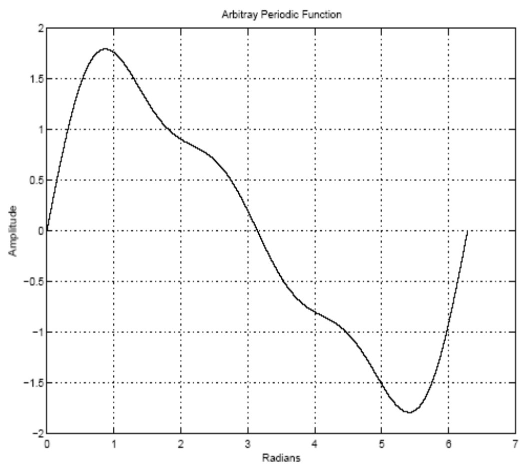

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Figure1.7.An arbitrary waveform created by summing up sin functions

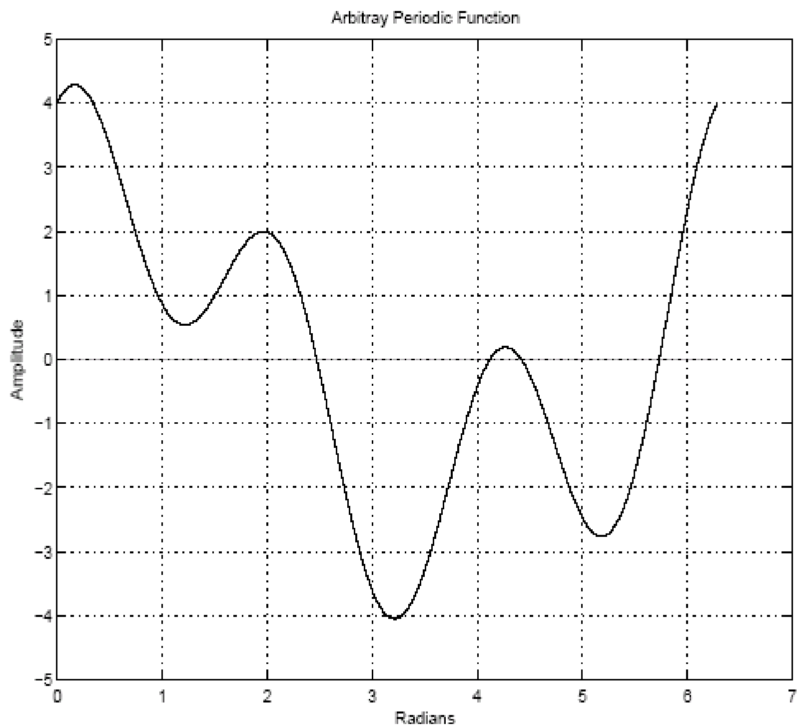
The other possibility of an arbitrary waveform is that it may begin with a non-zero height, indicating the presence of cosine waves in addition to the sin waves. As with the sin wave, the fundamental cosine function period must equal the period of the desired waveform. Also the subsequent cosine function periods must be integer multiples of the fundamental periods. We denote these cosine waves as $b_n \times \cos(n\omega t)$, where *b* is the amplitude or the coefficient of the cosine wave and ω is the frequency.

Using the sin and cosine functions an arbitrary waveform may be described as,

$$
\begin{aligned}
& f(t) = \\
& a_1 \sin(\omega t) + b_1 \cos(\omega t) + \\
& a_2 \sin(2\omega t) + b_2 \cos(2\omega t) + \\
& a_2 \sin(3\omega t) + b_2 \cos(3\omega t) + \\
& \vdots \\
& a_n \times \sin(n\omega t) + b_n \times \cos(n\omega t) \\
& = \sum_{n=1}^{\infty} a_n \sin(n\omega t) + \sum_{n=1}^{\infty} b_n \cos(n\omega t)
\end{aligned}
$$

For example, adding the following two cos functions to the arbitrary waveform of the Figure 1.7 we see a new pattern emerging as shown in Figure 1.8.

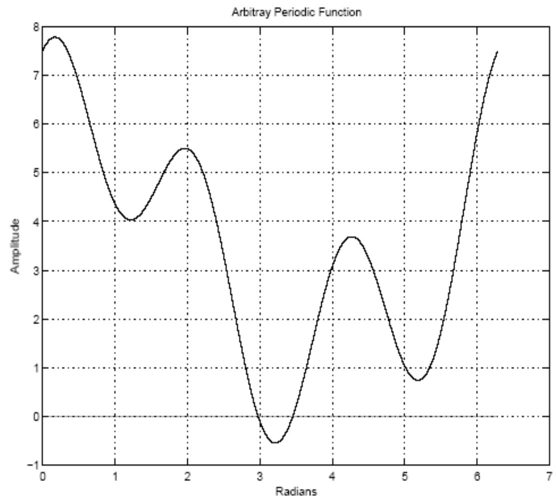
$$f(\omega) = 1.5\sin(t) + 0.5\sin(2t) + 0.3\sin(3t) + 2\cos(t) + 2\cos(3t)$$



******* Insert Figure 1.8 here******
Figure 1.8.Addition of cos waves to sin waves

A complete cycle of a sin and cosine wave includes a positive half and a negative equal half. In the previous example we saw how an arbitrary waveform was generated using only sin and cosine functions. **If we try to find the total area under the curve, it should add up to be equal to zero.** In an arbitrary waveform, a non-zero value for an area of one complete cycle indicates the presence of a **constant value** in addition to having sin and cosine waves. You could see the effect of adding a constant value of 3.5 to the arbitrary waveform of the previous example as shown in the Figure 1.9. The entire wave is almost shifted in the upper half of the axis. Now, we have a complete formula for an arbitrary function, there is a constant value (also called the **dc component** of the wave) and a series addition of sin and cos functions.

$$f(t) = 3.5 + 1.5\sin(t) + 0.5\sin(2t) + 0.3\sin(3t) + 2\cos(t) + 2\cos(3t)$$



******* Insert Figure 1.9 here******
Figure 1.9.Addition of constant value of 3.5 shifts the wave to the upper half of axis.

The complete formula of an arbitrary waveform is given as,

$$
\begin{aligned}
& f(t) = \\
& a_0 + \\
& a_1 \sin(\omega t) + b_1 \cos(\omega t) + \\
& a_2 \sin(2\omega t) + b_2 \cos(2\omega t) + \\
& a_2 \sin(3\omega t) + b_2 \cos(3\omega t) + \\
& \vdots \\
& a_n \sin(n\omega t) + b_n \cos(n\omega t) \\
& = a_0 + \sum_{n=1}^{\infty} a_n \sin(n\omega t) + \sum_{n=1}^{\infty} b_n \cos(n\omega t)
\end{aligned}
\qquad 1.1
$$

Where $f(t)$ is the desired waveform; a_0 is the **dc constant**, a_n are the **sin coefficient,** b_n are the **cosine coefficient,** ω is the angular frequency equal to the arbitrary waveform frequency and the n is an integer.

The Equation 1.1 can also be described in terms of frequency as

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin(n2\pi ft) + \sum_{n=1}^{\infty} b_n \cos(n2\pi ft)$$

Or if the period of one complete cycle is given as T then the Equation 1.1 becomes,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin(\frac{n2\pi}{T} t) + \sum_{n=1}^{\infty} b_n \cos(\frac{n2\pi}{T} t)$$

The two things we have determined so far in composing an arbitrary waveform are; that the fundamental frequency is same as the frequency of the arbitrary waveform and the fact that the subsequent frequencies are integer multiples of the fundamental frequency. The remaining two aspects are 1) determining the amplitudes 2) the total number of waves.

Fourier analysis

In this section we will answer the question of how to analyze an arbitrary waveform and determine its constituent frequencies.

Given an arbitrary function the task is,

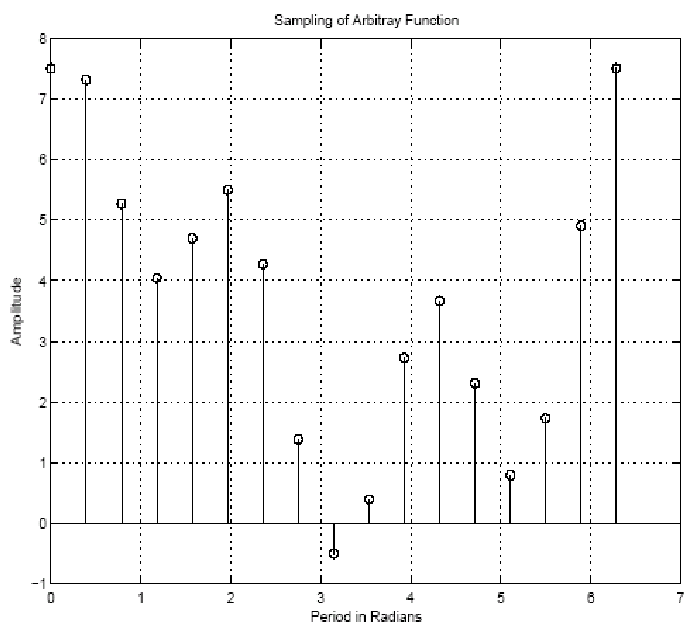
To find the starting or the **fundamental frequency** of the series that made the waveform.
To find the **total number of waves** in the series
To find the **coefficients** or the amplitudes of the sin and cos waves in the series.
To find the **dc-component** of the series

Example 1.1,

We proceed with the analysis of a physical event by using data that were acquired by a digital computer. The data acquisition rate was 16 samples per second and the data for one complete cycle are presented in the table 1.1. The graphical representation of data is shown in Figure 1.10. And this is the function we would like to analyze to find its component frequencies as described by Fourier.

Table 1.1 Data acquisition values of 16 events, at interval of 0.0625 sec apart

K	$k\Delta t$	$f(t)$
0	0	7.5
1	0.4	7.29
2	0.8	5.2
3	1.2	4.03
4	1.6	4.79
5	2	5.49
6	2.4	4
7	2.8	1.02
8	3.2	-0.55
9	3.6	0.76
10	4	3.08
11	4.4	3.54
12	4.8	1.86
13	5.2	0.74
14	5.6	2.43
15	6	5.83



******* Insert Figure 1.10 here******

Figure 1.10. The graph is showing the data acquisition values for one period.

When we use computers to process data it is no longer a continuous time processing. The time t is valid only at the instance when we have acquired the sample data. If k is the sample number and Δt is the time interval per sample then the discrete time variable equivalent of continuous time t is $k\Delta t$. Thus, the continuous time frequency function $\sin(\omega t)$ equals the discrete time frequency function $\sin(\omega k\Delta t)$. This can also be defined in terms of the period of the waveform as $\sin(2\pi k\Delta t/T)$.

Fundamental frequency

If T is the period of the arbitrary waveform then its fundamental constituent frequencies are $a_1 \sin(2\pi k\Delta t/T)$ and $b_1 \cos(2\pi k\Delta t/T)$. The integer multiples of fundamental frequencies are $a_n \sin(n2\pi k\Delta t/T)$ and $b_n \cos(n2\pi k\Delta t/T)$. This gives us the **Discrete Time Fourier Series** as,

$$
\begin{aligned}
&f(t) = \\
&a_0 + \\
&a_1 \sin(2\pi k\Delta t/T) + b_1 \cos(2\pi k\Delta t/T) + \\
&a_2 \sin(4\pi k\Delta t/T) + b_2 \cos(4\pi k\Delta t/T) + \\
&a_2 \sin(6\pi k\Delta t/T) + b_2 \cos(6\pi k\Delta t/T) + \\
&\vdots \\
&a_n \sin(n2\pi k\Delta t/T) + b_n \cos(n2\pi k\Delta t/T) \\
&= a_0 + \sum_{n=1}^{N} a_n \sin(n2\pi k\Delta t/T) + \sum_{n=1}^{N} b_n \cos(n2\pi k\Delta t/T)
\end{aligned}
$$

Where a_n b_n the coefficients and N is the total number of harmonics still need to be determined.

The previous example had 16 samples for one complete set of information that makes the fundamental period $T = 1\sec$ and the sampling interval Δt =1/16 or 0.0625 seconds per sample.

.

$k = 0..15$
$f = 1 cycle / \sec$
$\Delta t = 1/16 = 0.0625 \sec$
$T = 1 \sec$
$2\pi k \Delta t / T = 0.4k$

The fundamental constituent frequencies are

$a_1 \sin(0.4k)$, $b_1 \cos(0.4k)$

The discrete time Fourier series of the example arbitrary waveform is,

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \times \sin(n0.4k) + \sum_{n=1}^{N} b_n \times \cos(n0.4k)$$

Harmonics

Harmonics are the Integer multiples of the fundamental frequencies but the total number is different for the discrete time analysis from the continuous time analysis. You may have noticed in the section of Fourier synthesis that as we started adding more frequencies in to the series the picture of the original waveform started getting clearer. Ideally, for a continuous time function, it may require infinite frequencies to formulate an arbitrary waveform, especially if there are jump discontinuities in the function, this is Gibb's phenomenon and we will discuss it later. But for a discrete time function there is a limit to the number of frequencies that can be identified.

The number of frequencies in the arbitrary waveform

In discrete time processing the time t is valid only at the instance $k\Delta t$ when the sample is being acquired. It is important for the sample points to repeat the pattern in the subsequent cycles for the solution to be valid for periodic functions. If there are frequencies in the original waveform whose period is shorter then the sampling period Δt then it is quite possible that our sampling process would pick up these frequencies at the wrong time. The result would be totally unpredictable, just like the old western movies where wheels turn backward and the stagecoach moves forward. The camera could not take pictures fast enough. The number of spokes of the wheel passing through one point in one second was faster then the number of frames grabbed by the camera in one second.

To see the effect in real life, imagine there is only one frequency of 4 Hz in the input signal. If the sampling rate is 3 Hz and using that to reconstitute the original wave it would give us the answer of 6 Hz, a totally wrong answer for the original wave approximation.

We are essentially introducing **alias frequencies** into our system by not being able to process data fast enough. In practice, input frequencies are filtered out using hardware techniques before analog to digital conversion has taken place. There is a limit to the maximum frequency that a data acquisition system is capable of processing.

Nyquist frequency

Nyquist frequency is the limit beyond which we start introducing alias frequencies into our system. It is clear that a slow sampling rate may distort the reconstruction of the desired waveform using Fourier series. We need enough samples to determine at least one period of the wave and that number is 2 samples per period. This makes the number of frequencies in a waveform equals to half the number of samples in one cycle. With the same token the last frequency that can be recognized is half the sampling rate of the system. The last frequency is also called the Nyquest frequency of the system for a given sampling rate.

Referring back to the Example 1.1, the sampling rate of 16 samples per second limits us to the maximum input frequency of 8 Hz. With this limitation, we would reintroduce the Fourier series summation process with the number of frequencies N =8.

$$f(t) = a_0 + \sum_{n=1}^{8} a_n \times \sin(n0.4k) + \sum_{n=1}^{8} b_n \times \cos(n0.4k)$$

The continuous time function on the other hand have no upper limit, we may have to perform summation up to infinity for the series to have true representation of the waveform.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \times \sin(n2\pi ft) + \sum_{n=1}^{\infty} b_n \times \cos(n2\pi ft)$$

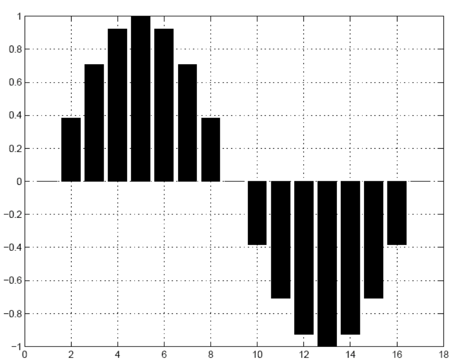
So far we have managed to complete two out of the four tasks for our example problem, a) The starting or the **fundamental frequency** is 1 Hz, and b) the **total number of frequencies** in the series N=8. Next is finding the amplitude.

Amplitude of the sin and cosine waves

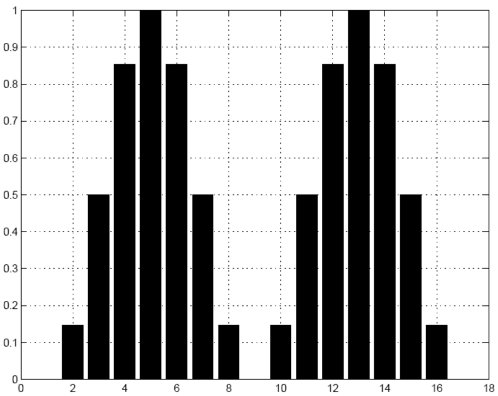
Now we discuss the difficult part of finding the amplitude of the individual component waves. The method is so brilliant that you cannot help but appreciate the genius in Fourier. His method has opened up a new branch of mathematics and in honor of his contribution the process is called finding **Fourier Coefficients** instead of finding the amplitude of the waves. The amplitudes of the sin waves is the Fourier sin coefficients and amplitudes of the cos waves is the Fourier cos coefficients. The method is based on the orthogonal property of the trigonometric functions.

Orthogonal functions

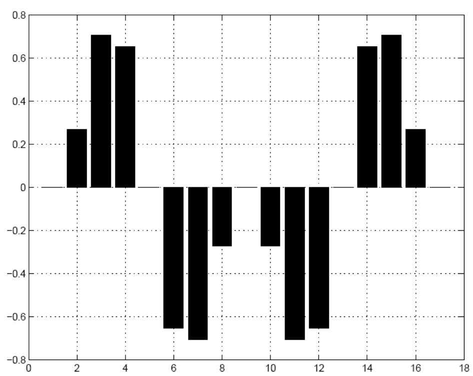
A complete cycle of sin and cosine function includes a positive half and negative half. The result of multiplying a sin function with another sin function would produce equally positive and negative values, but when multiplied to it there would be no negative value, as shown in Figure 1.11. If the result of multiplying a sin function by any other sin function is integrated for one complete cycle it should add up to be equal to zero, except when the frequency matches with the function itself. (Simply adding the little rectangles (see Figure 1.12) formed by multiplying the height and the length Δt could do the integration). Orthogonal functions are the functions that when multiplied and integrated produce zero area, except when they are multiplied by themselves, and trigonometric functions fall into this category.



******* Insert Figure 1.11.a here******

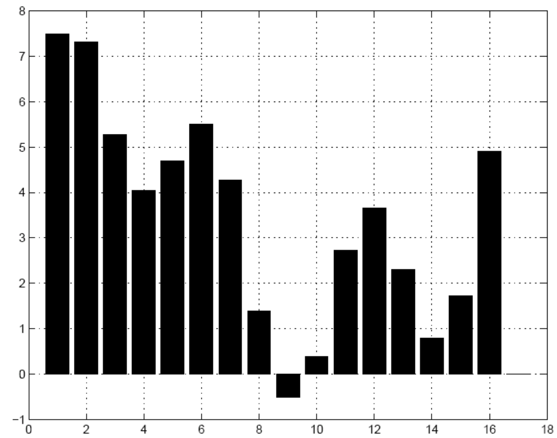


******* Insert Figure 1.11.b here******



******* Insert Figure 1.11.c here******
Figure 1.11.a) Area under the curve of sin function and b) sin squared, c) orthogonal property of the sin function

This is the property Fourier exploited to extract the sin and cosine coefficients for the trigonometric series of the arbitrary waveform.



******* Insert Figure 1.12 here******
Fig 1.12. Integrating the function using rectangular area method

Computing Fourier coefficient a_n and b_n

The square integral of the trigonometric function acts like a filter of frequency where only the matching frequency survive and the rest are vanished. You can see this graphically as shown in Figure 1.11. The function $a_n \sin(\omega t)$ was multiplied by $\sin(\omega t)$ and integrated for one complete cycle of period T. The area produced by the multiplication and summation was equal to half the area of the rectangle formed $height \times width$,

$$Area = \frac{a_n \times T}{2}$$

$$\frac{a_n \times T}{2} = \sum_{k=0}^{K-1} f(k) \times a_n \sin(n 2\pi k \Delta t / T) \times \Delta t$$

Thus, from the result of summing all the areas we could obtain the coefficient a_n as

$$a_n = \frac{2}{T} \sum_{k=0}^{K-1} f(k) \times a_n \sin(n 2\pi k \Delta t / T) \times \Delta t$$

Similarly, the cosine coefficients for discrete time processing are obtained as,

$$b_n = \frac{2}{T} \sum_{k=0}^{K-1} f(k) \times a_n \cos(n 2\pi k \Delta t / T) \times \Delta t$$

For a continuous time function we take a very small section of the width as *dt* and integrate the multiplication process for one complete cycle of 2π radians. The area produced by the multiplication and integration was equal to half the area of the rectangle formed $height \times width$,

$$Area = \frac{a_n \times 2\pi}{2}$$

And the coefficient a_n

$$a_n = \frac{1}{\pi} \int_{t=0}^{2\pi} f(t) \times a_n \sin(\omega_n t) \times dt$$

The cosine coefficients for continuous time processing,

$$b_n = \frac{1}{\pi} \int_{t=0}^{2\pi} f(t) \times a_n \cos(\omega_n t) \times dt$$

The process should be repeated for all the frequencies in the waveform up to the Nyquist frequency in the system for discrete time processing and up to infinity for continuous time processing.

Computing the dc constant a_0

We know the effect of the dc constant a_0. It shifts the entire wave either below or above the base line. Suppose we did not have the dc constant, the area under the curve of the complex wave would be exactly equal to 0 over one complete period. Any non-zero value for the area indicates contribution of the dc component to the complex wave. To find the area for one period *T*, we can simply add all the little rectangles we get by multiplying each sample point $f_k(k)$ with ΔT of sampling frequency, as shown in fig. 1.9.The constant a_0 is obtained by the dividing the area with the width, as height = Area/width.

Height = Area/Width

$$a_0 = \frac{\sum_{k=0}^{K-1} f_k(k) \times \Delta T}{T}$$

Similarly, for the continuous time processing the dc constant a_0 is computed as

$$a_0 = \frac{\int_0^{2\pi} f(t)dt}{2\pi}$$

This gives us a complete picture of Fourier analysis meaning, finding the constituent frequencies in an arbitrary waveform.

Discrete Fourier Series

The Fourier series of an arbitrary waveform obtained with discrete time sampling is described as,

$$f(k) = a_0 + \sum_{n=1}^{N} a_n \times \sin(n2\pi k\Delta t/T) + \sum_{n=1}^{N} b_n \times \cos(n2\pi k\Delta t/T) \qquad 1.2$$

N=Total number of samples / 2, (The Nyquist frequency)

k is the sample number

K=Total number of samples

T is period of the arbitrary waveform

The dc constant

$$a_0 = \frac{1}{T}\sum_{k=0}^{K-1} f_k(k) \times \Delta T \qquad 1.3$$

The sin coefficients

$$a_n = \frac{2}{T}\sum_{k=0}^{K-1} f(k) \times a_n \sin(n2\pi k\Delta t/T) \times \Delta t \qquad 1.4$$

The cosine coefficients

$$b_n = \frac{2}{T}\sum_{k=0}^{K-1} f(k) \times a_n \cos(n2\pi k\Delta t/T) \times \Delta t \qquad 1.5$$

Continuous Time Fourier Series

The Fourier series for a continuous time function is given as,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \times \sin(n2\pi\omega t) + \sum_{n=1}^{\infty} b_n \times \cos(n2\pi\omega t) \qquad 1.6$$

The dc constant

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) \times dt \qquad 1.7$$

The sin coefficients

$$a_n = \frac{1}{\pi} \int_{t=0}^{2\pi} f(t) \times a_n \sin(\omega_n t) \times dt \qquad 1.8$$

The cosine coefficients

$$b_n = \frac{1}{\pi} \int_{t=0}^{2\pi} f(t) \times a_n \cos(\omega_n t) \times dt \qquad 1.9$$

Referring back to the Example 1.1, the Table 1.2 shows the computation values for the fundamental frequency.

Table 1.2. a) Sample #k, b) The discrete time values of the function, c) The Function value multiplied by the sin, d) Area of the rectangles,

K	Function value $f(k)$	Sin multiplier $f(k) \times \sin(2\pi fk\Delta T / T)$	Area of rectangle $f(k) \times \sin(2\pi fk\Delta T / T)\Delta T$
0	7.5	0	0
1	7.32	2.8	0.175
2	5.27	3.73	0.233
3	4.04	3.73	0.233
4	4.7	4.7	0.294
5	5.5	5.08	0.318
6	4.27	3.02	0.189
7	1.38	0.53	0.033
8	-0.5	0	0

9	0.39	-0.15	-0.01
10	2.73	-1.93	-0.12
11	3.66	-3.39	-0.212
12	2.3	-2.3	-0.144
13	0.79	-0.73	-0.045
14	1.73	-1.22	-0.076
15	4.91	-1.88	-0.117
		$a_1 = \frac{2}{T}\sum_{k=0}^{K-1} f(k) \times \sin(2\pi fk\Delta T / T)\Delta T$	1.5

We obtained the following coefficients for rest of the harmonics n=1..8, using the discrete time Equations 1.2, 1.3, 1.4 and 1.5.

Fourier sin and coefficients

$$a_1 = 1.5 \ldots\ldots\ldots b_1 = 2$$
$$a_2 = 0.5 \ldots\ldots\ldots b_2 = 0$$
$$a_3 = 0.3 \ldots\ldots\ldots b_3 = 2$$
$$a_4 = 0 \ldots\ldots\ldots\ldots b_4 = 0$$
$$a_5 = 0 \ldots\ldots\ldots\ldots b_5 = 0$$
$$a_6 = 0 \ldots\ldots\ldots\ldots b_6 = 0$$
$$a_7 = 0 \ldots\ldots\ldots\ldots b_7 = 0$$
$$a_8 = 0 \ldots\ldots\ldots\ldots b_8 = 0$$

We get the following harmonics by substituting the computed Fourier coefficients into the Fourier series formula

Harmonics

$$1.5\sin(1 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots 2\cos(1 \times 2\pi k\Delta t)$$
$$0.5\sin(2 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots 0\cos(2 \times 2\pi k\Delta t)$$
$$0.3\sin(3 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots 2\cos(3 \times 2\pi k\Delta t)$$
$$0\sin(4 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots\ldots 0\cos(4 \times 2\pi k\Delta t)$$
$$0\sin(5 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots\ldots 0\cos(5 \times 2\pi k\Delta t)$$
$$0\sin(6 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots\ldots 0\cos(6 \times 2\pi k\Delta t)$$
$$0\sin(7 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots\ldots 0\cos(7 \times 2\pi k\Delta t)$$
$$0\sin(8 \times 2\pi k\Delta t) \ldots\ldots\ldots\ldots\ldots 0\cos(8 \times 2\pi k\Delta t)$$

The dc constant a_0

The constant a0 is obtained by dividing the total area with the total width.

Height = Area/Width

$$a_0 = \frac{\sum_{k=0}^{N-1} f_k(k\Delta T) \times \Delta T}{T} = 3.5$$

The Fourier series for the arbitrary waveform described by the data points of Example 1.1 may be obtained by substituting the Fourier coefficients and the harmonics as follows,

$$f(t) = 3.5 + 1.5\sin(t) + 0.5\sin(2t) + 0.3\sin(3t) + 2\cos(t) + 2\cos(3t)$$

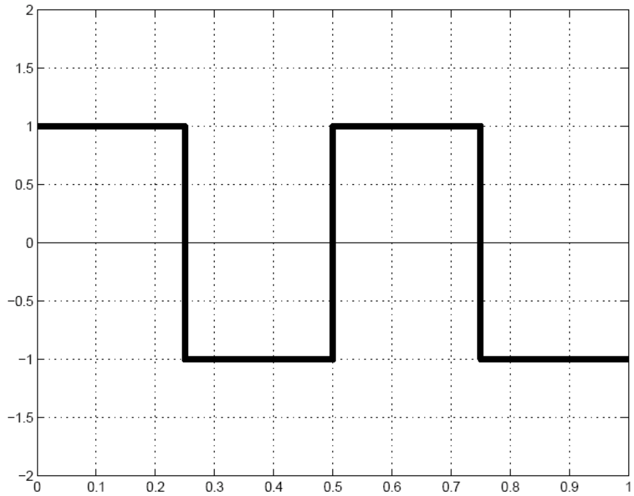
The process of Fourier analysis describes a function as a sum of series of sin and cosine frequencies and once the component frequencies are identified we may perform further analysis of the as if each was a separate input to the system.

We conclude the discussion of Fourier analysis with an example problem of a periodic disturbance in a physical system where Fourier analysis provides a logical solution.

Example 1.2.

A periodic force of 20 lb magnitude is applied on a system comprising a spring, mass and a dashpot as shown in Figure 1.13. Not knowing that hidden in the input force there are some frequencies that are close to the resonant or natural frequency of the system, a mysterious response will appear. The system will start oscillating with a frequency of 2 cycles per second while the input is only 1 cycle per second. We could have predicted this response of the system with the Fourier analysis of the input source, by identifying the component frequencies that are close to the resonant or natural frequency of the system.

First, identify the component frequencies present in the input force through Fourier analysis.



******* Insert Figure 1.13 here******
Fig 1.13. Integrating the function using rectangular area method

Computing the Fourier coefficients for the driving force:

The dc constant may be computed using Equation 1.7 but a glance at the driving force tells us that the upper halves and lower halves are equal, thus there is no dc component in the input force,

$$a_0 = 0$$

The cosine coefficients be computed using Equation 1.8, but the function begins at 0 and ends at 0 tells us there is no cosine component in the driving force,

$$b_n = 0$$

The sine coefficients are computed using the Equation 1.9, but the integral would cancel the positive area and the negative area. We can avoid the situation by integrating only half the area and multiplying it by 2,

$$a_n = \frac{2}{1/2}\int_0^{1/2} 20\sin(\frac{n\pi t}{1/2})dt$$

$$a_n = 80\left[-\frac{\cos 2n\pi t}{2n\pi}\right]_0^{1/2}$$

$$a_n = 40 \times \frac{1 - \cos n\pi}{n\pi} \qquad 1.10$$

For every even *n* in Equation 1.10 the coefficient value of $b_n = 0$

$$a_n = 0 \qquad n\ even$$

For odd n we get the following coefficient

$$a_n = \frac{80}{n\pi} \qquad n\ odd$$

Thus the component frequencies in the driving force are:

$$F(t) = \frac{80}{\pi}\left[\sin 2\pi t + \frac{\sin 6\pi t}{3} + \frac{\sin 10\pi t}{5} + \frac{\sin 14\pi t}{7} + \frac{\sin 18\pi t}{9}\cdots\right]$$

Each component frequency in the Fourier series contributes independently to the outcome of the system. For the sake of brevity we will compute the magnitude of the forces contributed by the first few terms only:

$$F_0(t) = \frac{80}{\pi}, F_1(t) = \frac{80}{3\pi}, F_2(t) = \frac{80}{5\pi}, F_3(t) = \frac{80}{7\pi}, F_4(t) = \frac{80}{9\pi}, \qquad 1.11$$

Systems response

The systems response is the natural response plus the forced response. We will analyze the natural response of the spring, mass and dashpot system of the Figure 1.13 with the help of Newton's law:

Force = mass x acceleration

We are interested in finding out if there is any component frequency in the input force that matches the natural response frequency of the system and if there is one, we can predict, the system will oscillate to the natural frequency with an amplified response. The total response will be the sum of the natural response and the forced response.

Assuming only the weight *w* of the mass m is significant and the acceleration due to vibration is only in the y direction the natural response could be obtained as follows,

$$mass = w / g \qquad acceleration = \frac{d^2 y}{dt^2}$$

The force due to gravity
$Fg = -w$
The force with which the spring exerts upon the weight
$Fk = w - ky$
The force due to viscous damping
$Fc = -c\dfrac{dy}{dt}$
Substituting all the forces into Newton's law

$$\frac{w}{g}\frac{d^2 y}{dt^2} = -w + (w - ky) - c\frac{dy}{dt}$$

After rearranging and summing all the forces, we obtain the natural response as,

$$\frac{w}{g}\frac{d^2 y}{dt^2} + c\frac{dy}{dt} + ky = 0 \qquad 1.12$$

The forces applied to the system of Equation 1.18 are given in the Fourier series of Equation 1.17. We could find the complete solution by computing the response due to each individual force and adding them all later, but there is a better solution.

If we could change the Equation 1.18 into a frequency dependent function (known as Transfer Function), then by simply plugging-in the frequencies and magnitudes of the Fourier series of Equation 1.17 we could analyze the system response. This method is being discussed in detail in Chapter 4 and 5 but for now we will proceed with the characteristic solution of the Equation 1.18 as follows,

The characteristic equation is:

$$\frac{w}{g}s^2 + cs + k = F_0 \qquad 1.13$$

Substituting the value for $s = j\omega$ in Equation 1.19 we get the output as complex valued function,

$$Y = \frac{F_0}{\frac{w}{g}(j\omega)^2 + cj\omega + k}$$

The magnitude as a function of frequency is obtained by separating real and imaginary components as follows,

$$Y = \frac{F_0}{\sqrt{(k - \frac{w}{g}\omega^2)^2 + (cj\omega)^2}}$$

And the phase response is,

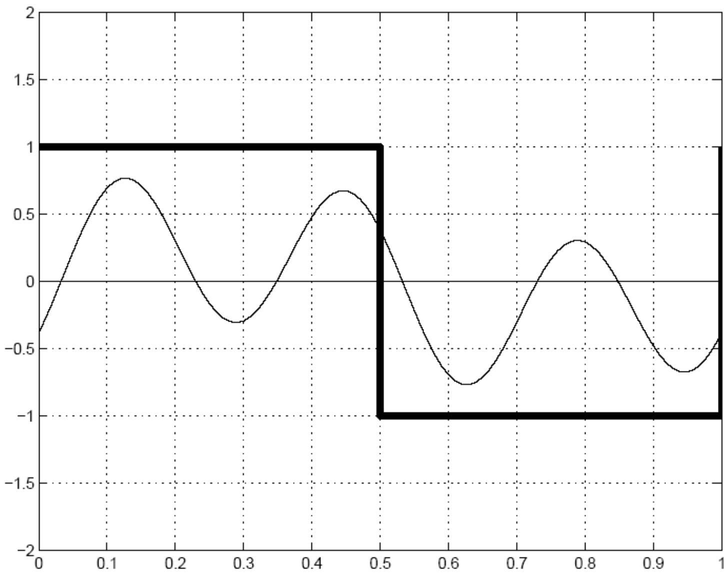
$$\phi = \tan^{-1}\frac{cj\omega}{k - \frac{w}{g}\omega^2}$$

For each input force and the frequency as defined in Equation 1.17 we could tabulate the output as shown in Table 1.3,

Table 1.3. Tabulation of the response of individual input frequencies for the system of Example 1.1.

Force applied	ω	$\frac{1}{\sqrt{(k - \frac{w}{g}\omega^2)^2 + (cj\omega)^2}}$	Phase $\phi = \tan^{-1}\frac{cj\omega}{k - \frac{w}{g}\omega^2}$	Response
$F_0 = \frac{80}{\pi}\sin(2\pi t)$	2π	.011	2	$0.28\sin(2\pi t - 2)$
$F_1 = \frac{80}{3\pi}\sin(\frac{2\pi t}{3})$	6π	.048	40	$0.58\sin(6\pi t - 40)$
$F_5 = \frac{80}{5\pi}\sin(\frac{2\pi t}{5})$	10π	.006	174	$0.03\sin(10\pi t - 174)$
$F_7 = \frac{80}{7\pi}\sin(\frac{2\pi t}{7})$	14π	.002	177	$0.01\sin(14\pi t - 177)$

The last column of Table 1.3 provides the response due to individual frequency components present in the input force applied to the system. A comparison of the magnitudes of the output response would indicate a dominant second harmonics in the input force (magnitude 0.58) and that explains why the example system started oscillating with twice the frequency of the input force as shown in the Figure 1.14.



******* Insert Figure 1.14 here******

Fig 1.14. Response to step function input for the Example 1.2

Summary

We have studied Fourier analysis as a method of describing an arbitrary function into its frequency components, a sum of series of sin and cosine functions with a dc component. The technique provides us with the capability of analyzing a physical system using a mathematical description. We have shown with an example that Fourier analysis can predict the response of a system that otherwise might be very difficult to obtain in time domain.

A continuous time function may be described by its Fourier Series as,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \times \sin(n2\pi\omega t) + \sum_{n=1}^{\infty} b_n \times \cos(n2\pi\omega t)$$

A discrete time function may be described by its Fourier Series as,

$$f(k) = a_0 + \sum_{n=1}^{N} a_n \times \sin(n2\pi k\Delta t/T) + \sum_{n=1}^{N} b_n \times \cos(n2\pi k\Delta t/T)$$